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The 2nd Nuclear Physics Tortoise Lecture Series (NPTLS) 2026

Pairing in ultracold atoms and neutron star

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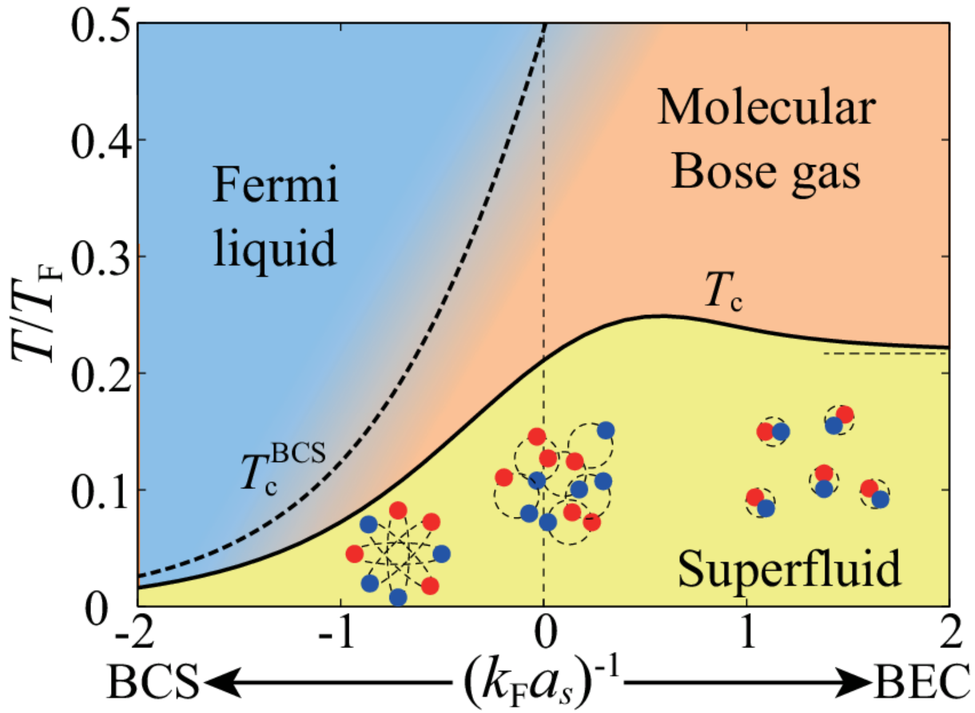
Part 2
-Ultracold Fermi gas and renormalization-

- Understanding the basics of BCS-BEC crossover



➡ Why is the horizontal axis the inverse scattering length a_s^{-1}?

Outline

- Theoretical model
- UV divergence and renormalization
- Physical interpretation of the scattering length
- Short summary

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- Theoretical model

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- Physical interpretation of the scattering length

- Short summary

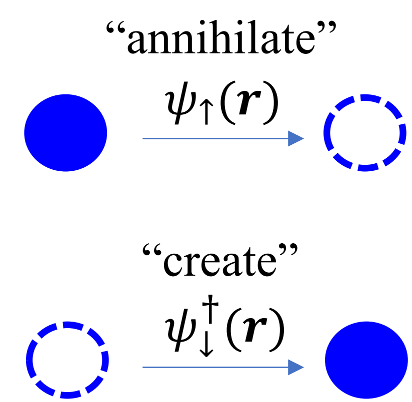
Model Hamiltonian

$$H = H_0 + V$$

We take $\hbar = k_{\mathrm{B}} = 1$

Kinetic term:

$$H_0 = \int d^3\boldsymbol{r} \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger(\boldsymbol{r}) \left(-\frac{\boldsymbol{\nabla}^2}{2m} - \mu + \mathcal{U}(\boldsymbol{r}) \right) \psi_\sigma(\boldsymbol{r})$$



σ: fermion spin ($\uparrow: +1/2\,, \downarrow: -1/2$)
m: fermion mass
$\psi_\sigma(\boldsymbol{r})$: fermion field operator
μ: fermion chemical potential
$\mathcal{U}(\boldsymbol{r})$: single-particle potential

*In ultracold atoms, σ represents two hyperfine states

Interaction term:

$$V = \frac{1}{2} \sum_{\sigma,\sigma'} \int d^3\boldsymbol{r} \int d^3\boldsymbol{r}'\, \psi_\sigma^\dagger(\boldsymbol{r}) \psi_{\sigma'}^\dagger(\boldsymbol{r}') U_{\sigma\sigma'}(\boldsymbol{r} - \boldsymbol{r}')\, \psi_{\sigma'}(\boldsymbol{r}') \psi_\sigma(\boldsymbol{r})$$

Contact-type interaction

Interaction term:

$$V = \frac{1}{2}\sum_{\sigma,\sigma'}\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'\,\psi_\sigma^\dagger(\boldsymbol{r})\psi_{\sigma'}^\dagger(\boldsymbol{r}')U_{\sigma\sigma'}(\boldsymbol{r}-\boldsymbol{r}')\,\psi_{\sigma'}(\boldsymbol{r}')\psi_\sigma(\boldsymbol{r})$$

Contact-type interaction

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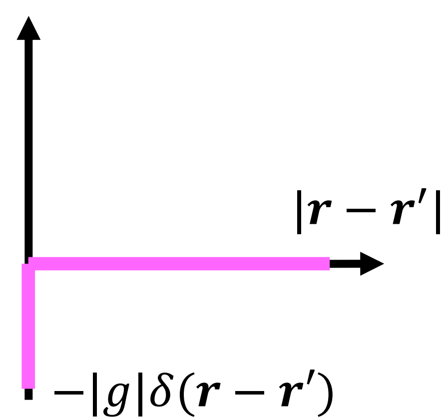
$g < 0$: attractive coupling strength $\boxed{U_{\sigma\sigma'}(\boldsymbol{r}-\boldsymbol{r}') \simeq g\delta(\boldsymbol{r}-\boldsymbol{r}')}$

$$V = \frac{g}{2}\sum_{\sigma,\sigma'}\int d^3\boldsymbol{r}\,\psi^\dagger_\sigma(\boldsymbol{r})\psi^\dagger_{\sigma'}(\boldsymbol{r})\,\psi_{\sigma'}(\boldsymbol{r})\psi_\sigma(\boldsymbol{r})$$

Short-range attraction



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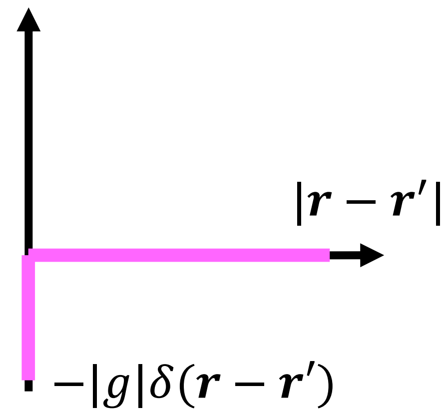
$g < 0$: attractive coupling strength

$$\boxed{U_{\sigma\sigma'}(\boldsymbol{r}-\boldsymbol{r}') \simeq g\delta(\boldsymbol{r}-\boldsymbol{r}')}$$

$$V = \frac{g}{2}\sum_{\sigma,\sigma'}\int d^3\boldsymbol{r}\,\psi_\sigma^\dagger(\boldsymbol{r})\psi_{\sigma'}^\dagger(\boldsymbol{r})\,\psi_{\sigma'}(\boldsymbol{r})\psi_\sigma(\boldsymbol{r})$$

Short-range attraction



Anti-commutation of fermion operator

$$\{\psi_\sigma(\boldsymbol{r}),\psi_\sigma(\boldsymbol{r})\} = 2\psi_\sigma(\boldsymbol{r})\psi_\sigma(\boldsymbol{r}) = 0$$

$$\left\{\psi_\sigma^\dagger(\boldsymbol{r}),\psi_\sigma^\dagger(\boldsymbol{r})\right\} = 2\psi_\sigma^\dagger(\boldsymbol{r})\psi_\sigma^\dagger(\boldsymbol{r}) = 0$$

$$\boxed{V = g\int d^3\boldsymbol{r}\,\psi_\uparrow^\dagger(\boldsymbol{r})\psi_\downarrow^\dagger(\boldsymbol{r})\psi_\downarrow(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})}$$

Fock-state representation

Expanding the field operators with respect to eigenstate of non-interacting Hamiltonian

$$\psi_\sigma(\boldsymbol{r}) = \sum_\alpha \Psi_\alpha(\boldsymbol{r}) c_{\alpha\sigma} \qquad \psi_\sigma^\dagger(\boldsymbol{r}) = \sum_\alpha \Psi_\alpha^*(\boldsymbol{r}) c_{\alpha\sigma}^\dagger$$

$\Psi_\alpha(\boldsymbol{r})$:eigenstate $$\left(-\frac{\boldsymbol{\nabla}^2}{2m} + \mathcal{U}(\boldsymbol{r})\right)\Psi_\alpha(\boldsymbol{r}) = E_\alpha \Psi_\alpha(\boldsymbol{r})$$

$c_{\alpha\sigma}$: fermion annihilation operator

$c_{\alpha\sigma}^\dagger$: fermion creation operator

Uniform case: $\mathcal{U}(\boldsymbol{r}) = 0$

$\alpha \to \boldsymbol{k}$ (momentum) $\Psi_\alpha(\boldsymbol{r}) \to \dfrac{1}{\sqrt{L^3}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ (plane wave)

L: system size

Momentum representation

Equivalent to the Fourier transformation

$$\psi_\sigma(\boldsymbol{r}) = \frac{1}{\sqrt{L^3}}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} c_{\boldsymbol{k}\sigma} \qquad \psi_\sigma^\dagger(\boldsymbol{r}) = \frac{1}{\sqrt{L^3}}\sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} c_{\boldsymbol{k}\sigma}^\dagger$$

Kinetic term: $H_0 = \int d^3\boldsymbol{r} \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger(\boldsymbol{r}) \left(-\frac{\boldsymbol{\nabla}^2}{2m} - \mu\right) \psi_\sigma(\boldsymbol{r})$

$$= \frac{1}{L^3}\int d^3\boldsymbol{r} \sum_{\sigma=\uparrow,\downarrow} \sum_{\boldsymbol{k},\boldsymbol{k}'} e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\, c_{\boldsymbol{k}\sigma}^\dagger \left(\frac{k^2}{2m} - \mu\right) c_{\boldsymbol{k}\sigma}$$

$$= \sum_{\boldsymbol{k},\sigma} \xi_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^\dagger c_{\boldsymbol{k}\sigma}$$

$$* \frac{1}{L^3}\int d^3 r\, e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = \delta_{\boldsymbol{k},\boldsymbol{k}'}$$

$\xi_{\boldsymbol{k}} = \frac{k^2}{2m} - \mu$: single-particle kinetic energy measured from μ

Momentum representation

Equivalent to the Fourier transformation

$$\psi_\sigma(\boldsymbol{r}) = \frac{1}{\sqrt{L^3}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} c_{\boldsymbol{k}\sigma} \qquad \psi_\sigma^\dagger(\boldsymbol{r}) = \frac{1}{\sqrt{L^3}} \sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} c_{\boldsymbol{k}\sigma}^\dagger$$

Kinetic term: $H_0 = \sum_{\boldsymbol{k},\sigma} \xi_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^\dagger c_{\boldsymbol{k}\sigma}$

Interaction term: $V = g \int d^3\boldsymbol{r}\, \psi_\uparrow^\dagger(\boldsymbol{r}) \psi_\downarrow^\dagger(\boldsymbol{r}) \psi_\downarrow(\boldsymbol{r}) \psi_\uparrow(\boldsymbol{r})$

$$= g \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4} \int d^3\boldsymbol{r}\, e^{-i(b_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\cdot\boldsymbol{r}} c_{\boldsymbol{k}_1\uparrow}^\dagger c_{\boldsymbol{k}_2\downarrow}^\dagger c_{\boldsymbol{k}_3\downarrow} c_{\boldsymbol{k}_4\uparrow}$$

$$= g \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} c_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow}^\dagger c_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow}^\dagger c_{-\boldsymbol{k}'+\boldsymbol{P}/2\downarrow} c_{\boldsymbol{k}'+\boldsymbol{P}/2\uparrow}$$

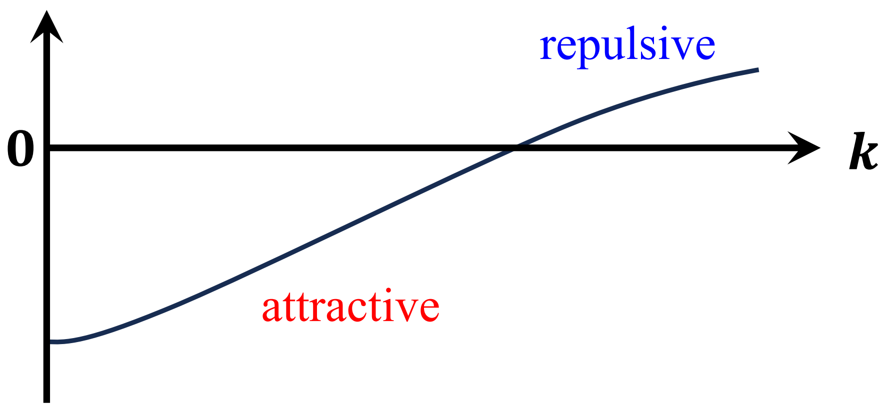
$\boldsymbol{k}$ ($\boldsymbol{k}'$): outgoing (incoming) relative momentum

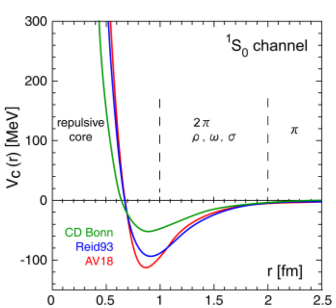
$\boldsymbol{P}$: center-of-mass momentum of two particles

Simple and valid Hamiltonian, but…

$$H = \sum_{\boldsymbol{k},\sigma} \xi_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + g \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} c^{\dagger}_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow} c^{\dagger}_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow} c_{-\boldsymbol{k}'+\boldsymbol{P}/2\downarrow} c_{\boldsymbol{k}'+\boldsymbol{P}/2\uparrow}$$

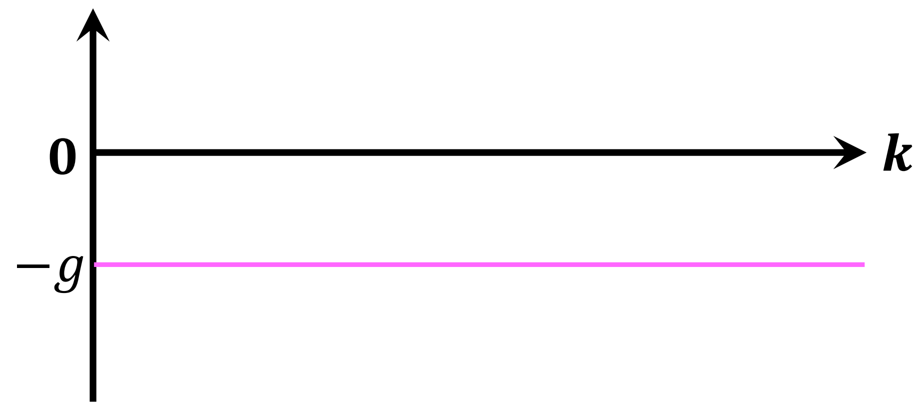
Typical nuclear force (momentum space)





PRL **99,** 022001 (2007).

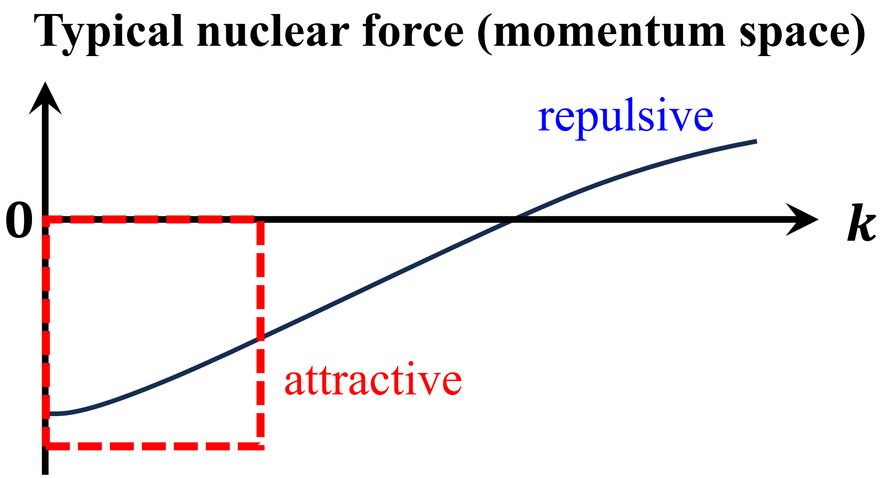
Contact-type interaction

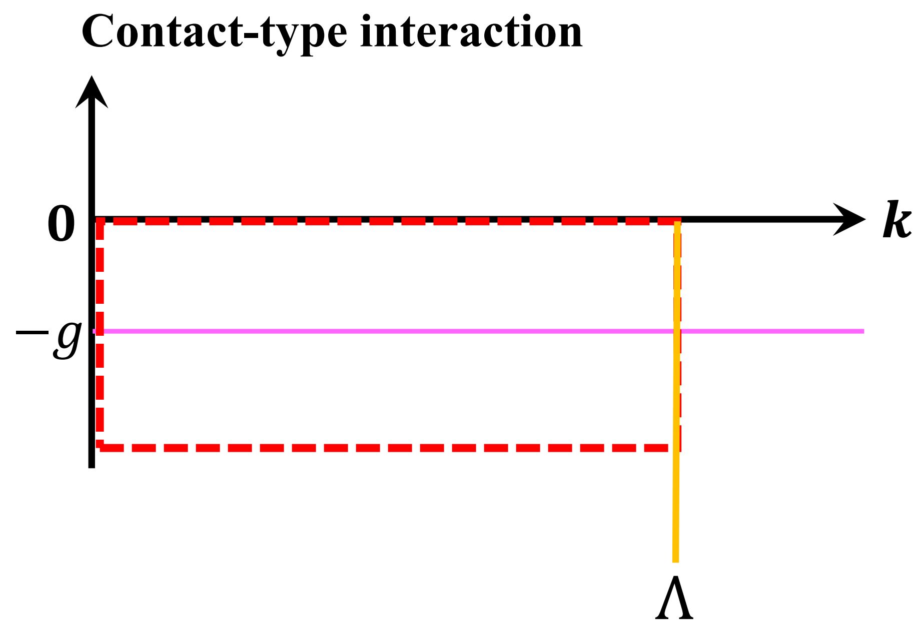


Attractive interaction works at any high relative momenta

➡Ultraviolet divergence!

Renormalization of coupling constant

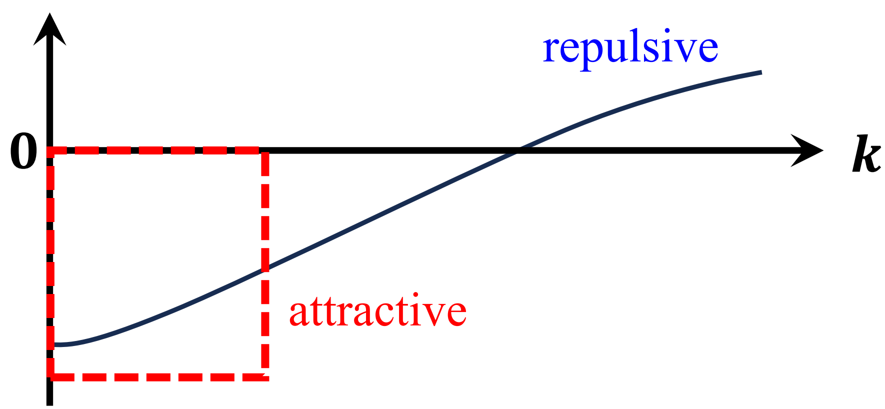




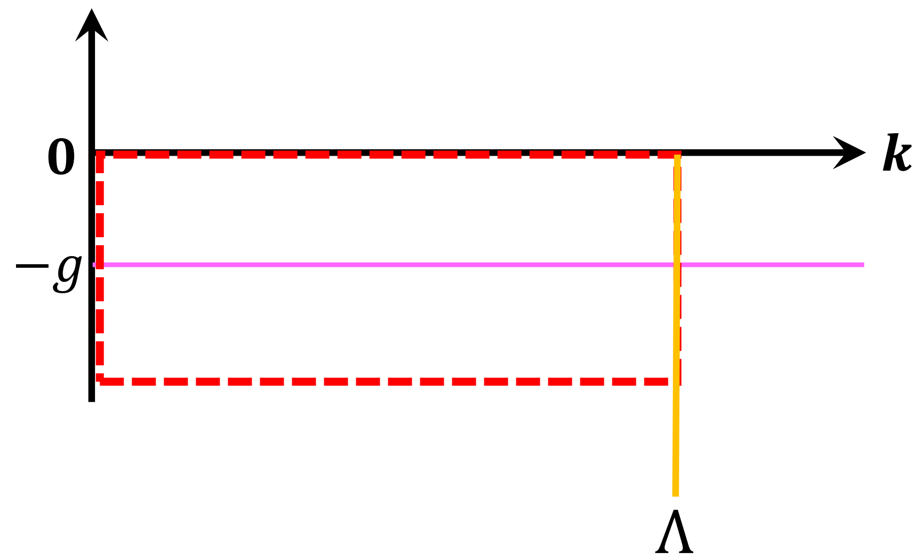
To avoid the UV divergence, we introduce a **momentum cutoff** Λ

Renormalization of coupling constant

Typical nuclear force (momentum space)



Contact-type interaction

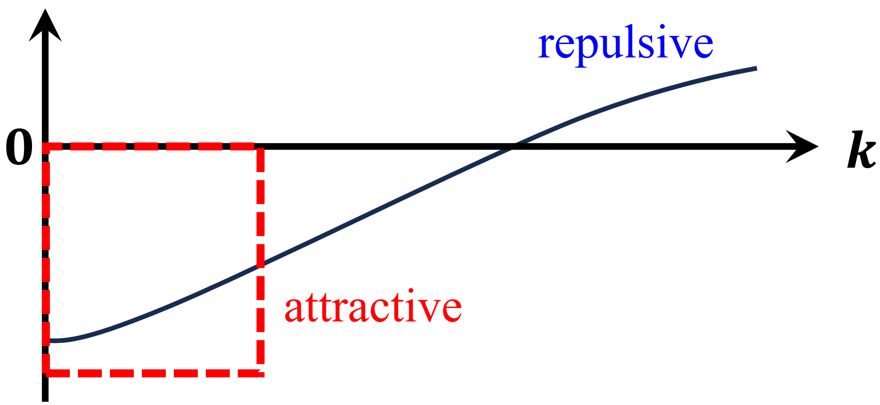


To avoid the UV divergence, we introduce a **momentum cutoff** Λ

Λ could be determined by short-range physics (e.g., repulsive core, effective range)

Renormalization of coupling constant

Typical nuclear force (momentum space)



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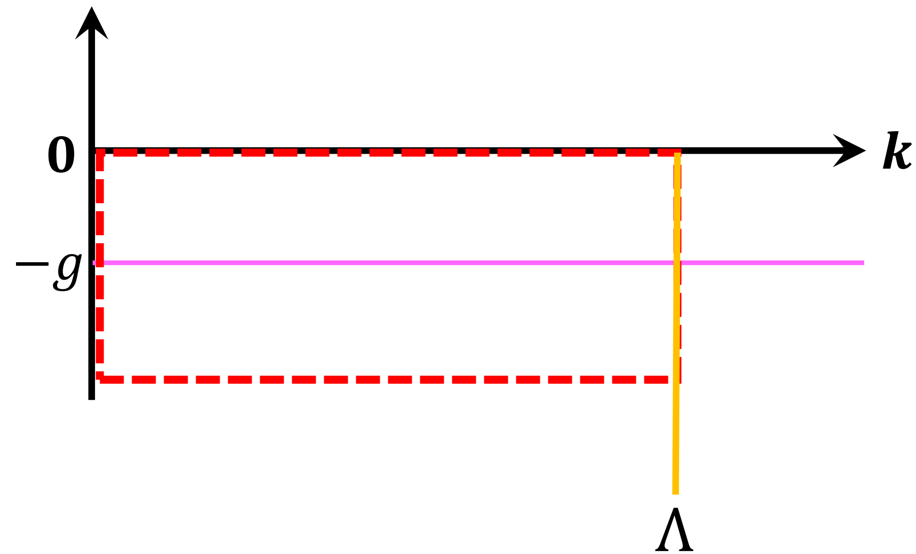


To avoid the UV divergence, we introduce a **momentum cutoff** Λ

Λ could be determined by short-range physics (e.g., repulsive core, effective range)

At sufficiently low energy system, physical quantities should not depend on Λ

e.g., *S*-wave scattering length a_s

Two-body scattering ($\mu = 0$)

Two-particle state with zero CoM momentum *P*=0

$$|\phi_{\boldsymbol{k}}\rangle = c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow}|0\rangle$$

After repeated two-body scattering (perturbation theory but infinite sum)

$$|\psi_{\boldsymbol{k}}\rangle = T_t \exp\left[-i\int_{-\infty}^{0} e^{\delta t}\hat{V}(t)dt\right]|\phi_{\boldsymbol{k}}\rangle$$

T_t: time-order product

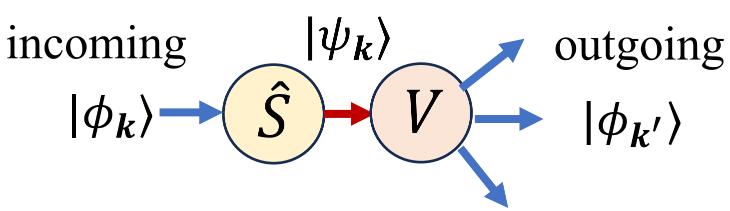
$\hat{V}(t) = e^{iH_0 t} V e^{-iH_0 t}$: interaction representation of V

δ: infinitesimal positive number for adiabatic switch-on

Scattering amplitude

$$f(\boldsymbol{k}, \boldsymbol{k}') = -\frac{m}{4\pi}\langle\phi_{\boldsymbol{k}'}|V|\psi_{\boldsymbol{k}}\rangle$$

(elastic: $|\boldsymbol{k}| = |\boldsymbol{k}'|$)



Contact-type interaction at $P = 0$

$$V = g \sum_{\boldsymbol{k},\boldsymbol{k}'} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow}$$

Scattering amplitude

$$f(\boldsymbol{k},\boldsymbol{k}') = -\frac{m}{4\pi}\langle \phi_{\boldsymbol{k}'}|V|\psi_{\boldsymbol{k}}\rangle = -\frac{m}{4\pi}\left[\langle \phi_{\boldsymbol{k}'}|V|\phi_{\boldsymbol{k}}\rangle - i\int_{-\infty}^{0} dt \langle \phi_{\boldsymbol{k}'}\middle|V\ \hat{V}(t)\middle|\phi_{\boldsymbol{k}}\rangle + \cdots\right]$$

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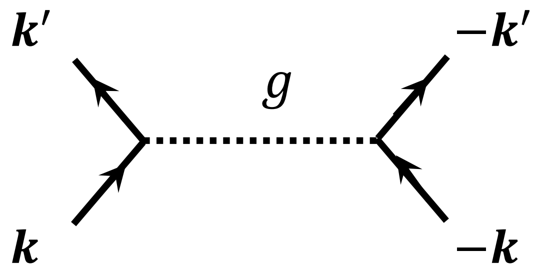
1st order perturbation with respect to *g*

$$\langle \phi_{\boldsymbol{k}'}|V|\phi_{\boldsymbol{k}}\rangle = \left\langle 0 \left| c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} V c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} \right| 0 \right\rangle = g$$



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Scattering amplitude

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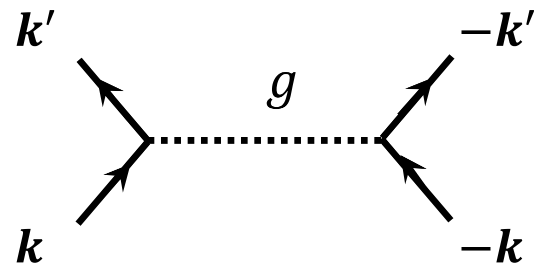
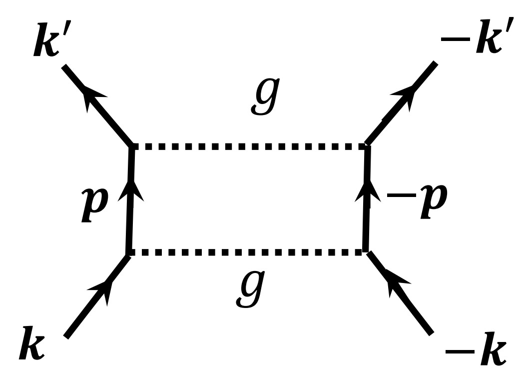


2nd order perturbation with respect to *g*

$$-i\int_{-\infty}^{0} dt \langle \phi_{\boldsymbol{k}'}\middle|V\,\hat{V}(t)\middle|\phi_{\boldsymbol{k}}\rangle$$

$$= -ig^2\int_{-\infty}^{0} dt\, e^{\delta t} \sum_{\boldsymbol{p}} \langle 0| c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} c^{\dagger}_{\boldsymbol{k}'\uparrow} c^{\dagger}_{-\boldsymbol{k}'\downarrow} c_{-\boldsymbol{p}\downarrow} c_{\boldsymbol{p}\uparrow} e^{iH_0 t} c^{\dagger}_{\boldsymbol{p}\uparrow} c^{\dagger}_{-\boldsymbol{p}\downarrow} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} e^{iH_0 t} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} |0\rangle$$



Contact-type interaction at $P = 0$

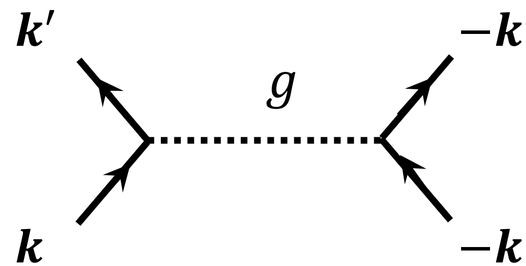
$$V = g \sum_{\boldsymbol{k},\boldsymbol{k}'} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow}$$

Scattering amplitude

$$f(\boldsymbol{k},\boldsymbol{k}') = -\frac{m}{4\pi}\langle\phi_{\boldsymbol{k}'}|V|\psi_{\boldsymbol{k}}\rangle = -\frac{m}{4\pi}\left[\langle\phi_{\boldsymbol{k}'}|V|\phi_{\boldsymbol{k}}\rangle - i\int_{-\infty}^{0} dt \langle\phi_{\boldsymbol{k}'}\middle|V\,\hat{V}(t)\middle|\phi_{\boldsymbol{k}}\rangle + \cdots\right]$$

1st order perturbation with respect to *g*

$$\langle\phi_{\boldsymbol{k}'}|V|\phi_{\boldsymbol{k}}\rangle = \left\langle 0 \middle| c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} V c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} \middle| 0 \right\rangle = g$$



2nd order perturbation with respect to *g*

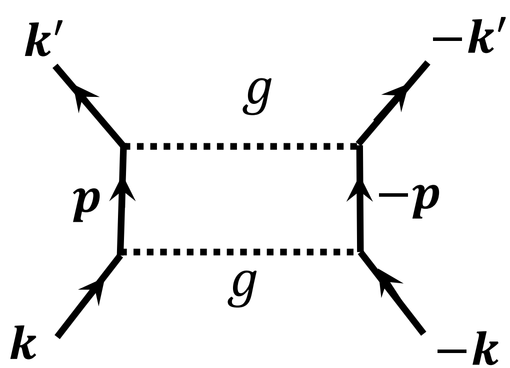
$$-i\int_{-\infty}^{0} dt \langle\phi_{\boldsymbol{k}'}\middle|V\,\hat{V}(t)\middle|\phi_{\boldsymbol{k}}\rangle$$

$$= -ig^2 \int_{-\infty}^{0} dt\, e^{\delta t} \sum_{\boldsymbol{p}} \langle 0| c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} c^\dagger_{\boldsymbol{k}'\uparrow} c^\dagger_{-\boldsymbol{k}'\downarrow} c_{-\boldsymbol{p}\downarrow} c_{\boldsymbol{p}\uparrow} e^{iH_0 t} c^\dagger_{\boldsymbol{p}\uparrow} c^\dagger_{-\boldsymbol{p}\downarrow} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} e^{iH_0 t} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} |0\rangle$$

$$= -ig^2 \sum_{\boldsymbol{p}} \int_{-\infty}^{0} dt\, e^{\delta t + i(2\varepsilon_{\boldsymbol{p}} - 2\varepsilon_{\boldsymbol{k}})t} = -g^2 \sum_{\boldsymbol{p}} \frac{1}{2\varepsilon_{\boldsymbol{k}} - 2\varepsilon_{\boldsymbol{p}} + i\delta}$$

$$H_0|\phi_{\boldsymbol{k}}\rangle = 2\varepsilon_{\boldsymbol{k}}|\phi_{\boldsymbol{k}}\rangle$$

$\varepsilon_{\boldsymbol{k}} = \frac{k^2}{2m}$: single-particle kinetic energy



Contact-type interaction at $P = 0$

$$V = g \sum_{\boldsymbol{k},\boldsymbol{k}'} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow}$$

Scattering amplitude

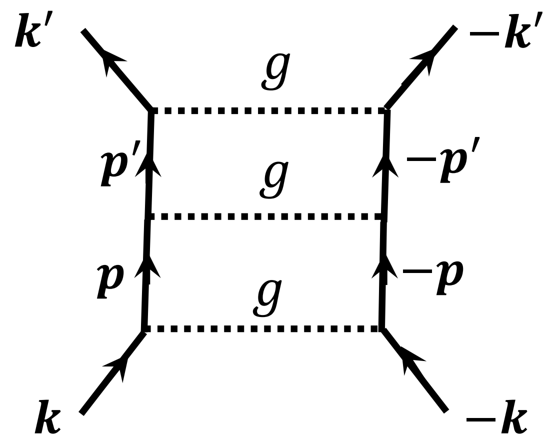
$$f(\boldsymbol{k},\boldsymbol{k}') = -\frac{m}{4\pi}\langle \phi_{\boldsymbol{k}'}|V|\psi_{\boldsymbol{k}}\rangle = -\frac{m}{4\pi}\left[\langle \phi_{\boldsymbol{k}'}|V|\phi_{\boldsymbol{k}}\rangle - i\int_{-\infty}^{0} dt \langle \phi_{\boldsymbol{k}'}|V\,\hat{V}(t)|\phi_{\boldsymbol{k}}\rangle + \cdots\right]$$

3rd order perturbation with respect to *g*

$$\frac{(-i)^2}{2!}\int_{-\infty}^{0} dt_a \int_{-\infty}^{0} dt_b e^{\delta(t_a+t_b)} \langle 0|T_t V\hat{V}(t_a)\hat{V}(t_b)|0\rangle$$

$$= (-i)^2 \int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} dt_2 e^{\delta(t_1+t_2)} \langle 0|V\hat{V}(t_1)\hat{V}(t_2)|0\rangle$$

$$T_t[V(t_a)V(t_b)] = \begin{cases} \hat{V}(t_a)\hat{V}(t_b) \equiv \hat{V}(t_1)\hat{V}(t_2) & t_a > t_b \\ V(t_b)V(t_a) \equiv \hat{V}(t_1)\hat{V}(t_2) & t_b > t_a \end{cases}$$

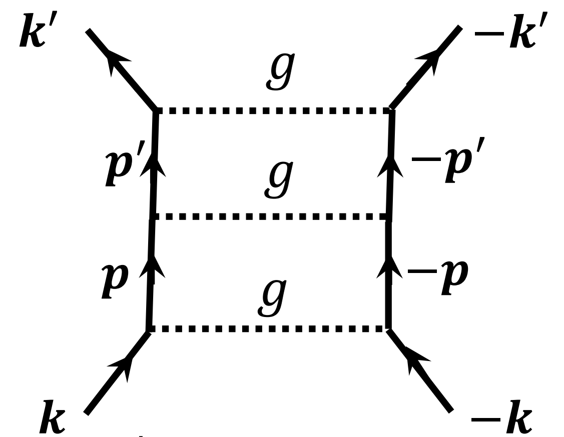


Contact-type interaction at $P = 0$

$$V = g \sum_{\boldsymbol{k},\boldsymbol{k}'} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow}$$

Scattering amplitude

$$f(\boldsymbol{k},\boldsymbol{k}') = -\frac{m}{4\pi}\langle \phi_{\boldsymbol{k}'}|V|\psi_{\boldsymbol{k}}\rangle = -\frac{m}{4\pi}\left[\langle \phi_{\boldsymbol{k}'}|V|\phi_{\boldsymbol{k}}\rangle - i\int_{-\infty}^{0} dt \langle \phi_{\boldsymbol{k}'}\middle|V\,\hat{V}(t)\middle|\phi_{\boldsymbol{k}}\rangle + \cdots\right]$$



3rd order perturbation with respect to g

$$\frac{(-i)^2}{2!}\int_{-\infty}^{0} dt_a \int_{-\infty}^{0} dt_b e^{\delta(t_a+t_b)}\,\langle 0|T_t V\hat{V}(t_a)\hat{V}(t_b)|0\rangle$$

$$= (-i)^2 \int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} dt_2 e^{\delta(t_1+t_2)}\,\langle 0|V\hat{V}(t_1)\hat{V}(t_2)|0\rangle$$

$$= (-i)^2 g^3 \int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} dt_2 e^{\delta(t_1+t_2)} \sum_{\boldsymbol{p},\boldsymbol{p}'} \langle 0|c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} c^{\dagger}_{\boldsymbol{k}'\uparrow} c^{\dagger}_{-\boldsymbol{k}'\downarrow} c_{-\boldsymbol{p}\downarrow} c_{\boldsymbol{p}\uparrow} e^{iH_0 t}$$

$$\times\, c^{\dagger}_{\boldsymbol{p}\uparrow} c^{\dagger}_{-\boldsymbol{p}\downarrow} c_{-\boldsymbol{p}'\downarrow} c_{\boldsymbol{p}'\uparrow} e^{iH_0 t} c^{\dagger}_{\boldsymbol{p}'\uparrow} c^{\dagger}_{-\boldsymbol{p}'\downarrow} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} e^{iH_0 t} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow}|0\rangle$$

$$= (-i)^2 g^3 \int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} dt_2 e^{\delta(t_1+t_2)} \sum_{\boldsymbol{p},\boldsymbol{p}'} e^{i\left(2\varepsilon_{\boldsymbol{p}}-2\varepsilon_{\boldsymbol{p}'}\right)t_1 + i\left(2\varepsilon_{\boldsymbol{p}'}-2\varepsilon_{\boldsymbol{k}}\right)t_1}$$

$$= g^3 \sum_{\boldsymbol{p}} \frac{1}{2\varepsilon_{\boldsymbol{k}} - 2\varepsilon_{\boldsymbol{p}} + i\delta} \sum_{\boldsymbol{p}'} \frac{1}{2\varepsilon_{\boldsymbol{k}} - 2\varepsilon_{\boldsymbol{p}'} + i\delta}$$

Contact-type interaction at $P = 0$

$$V = g \sum_{\boldsymbol{k},\boldsymbol{k}'} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow}$$

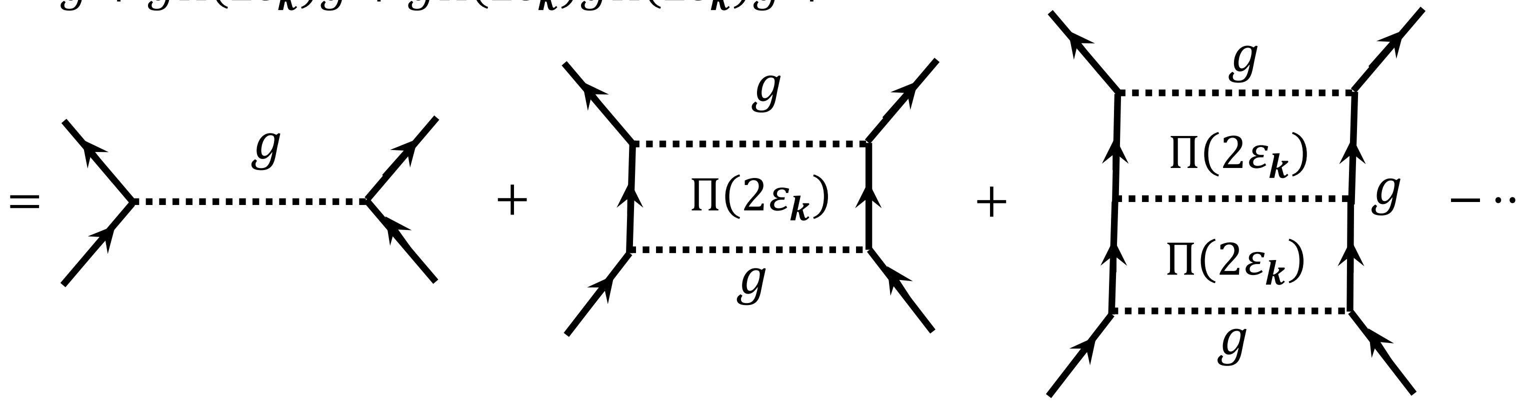
Retarded pair propagator

$$\Pi(2\varepsilon_{\boldsymbol{k}}) = \sum_{\boldsymbol{p}} \frac{1}{2\varepsilon_{\boldsymbol{k}} - 2\varepsilon_{\boldsymbol{p}} + i\delta}$$

Scattering amplitude

$$f(\boldsymbol{k},\boldsymbol{k}') = -\frac{m}{4\pi}\langle \phi_{\boldsymbol{k}'}|V|\psi_{\boldsymbol{k}}\rangle = -\frac{m}{4\pi}\left[\langle \phi_{\boldsymbol{k}'}|V|\phi_{\boldsymbol{k}}\rangle - i\int_{-\infty}^{0} dt \langle \phi_{\boldsymbol{k}'}|V\,\hat{V}(t)|\phi_{\boldsymbol{k}}\rangle + \cdots\right]$$

$$\langle \phi_{\boldsymbol{k}'}|V|\psi_{\boldsymbol{k}}\rangle = g + g\Pi(2\varepsilon_{\boldsymbol{k}})g + g\Pi(2\varepsilon_{\boldsymbol{k}})g\Pi(2\varepsilon_{\boldsymbol{k}})g + \cdots$$



Contact-type interaction at $P = 0$

$$V = g \sum_{\boldsymbol{k},\boldsymbol{k}'} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow}$$

Retarded pair propagator

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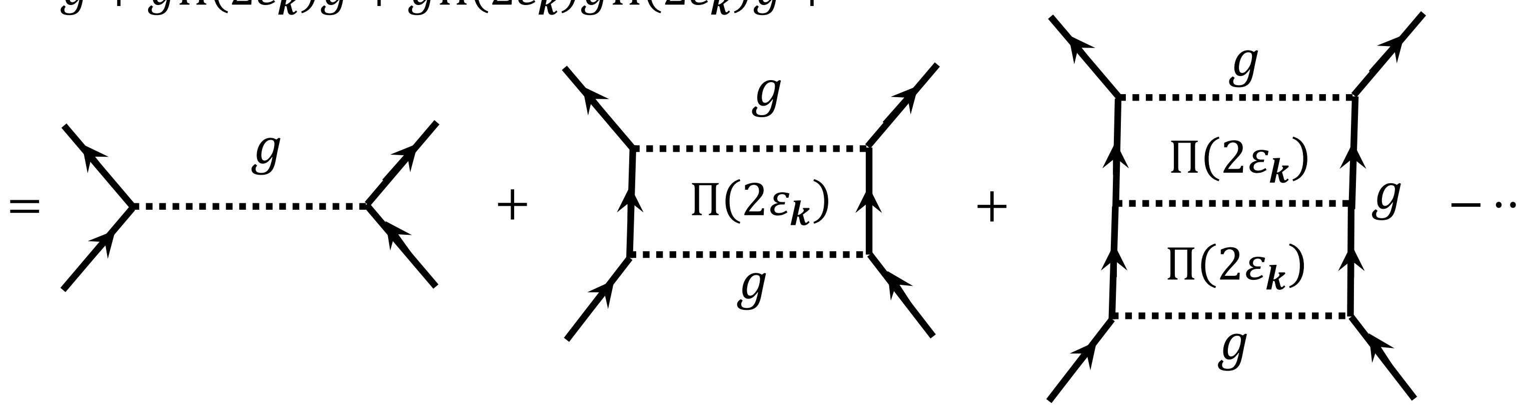
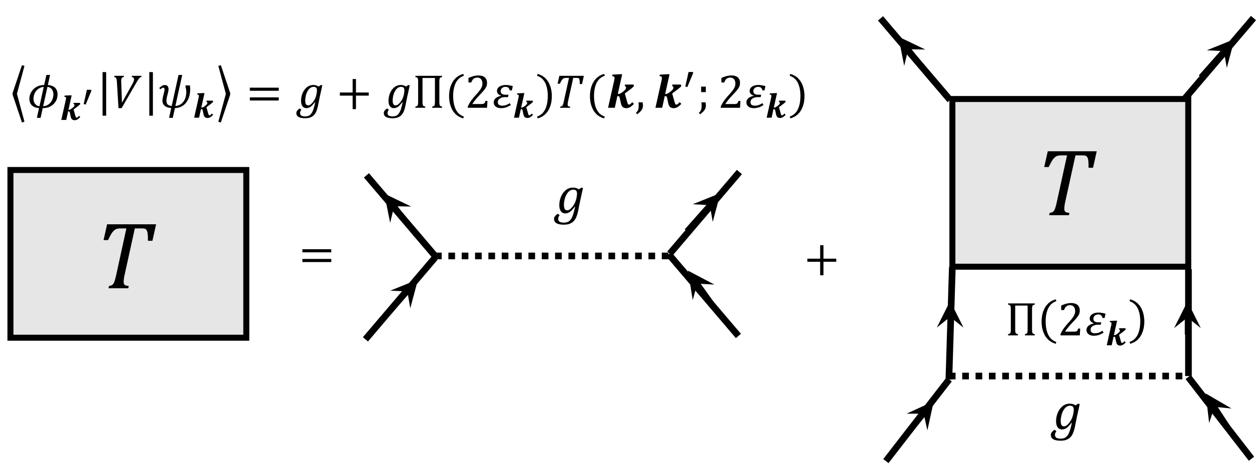


T matrix

$$T(\boldsymbol{k},\boldsymbol{k}';2\varepsilon_{\boldsymbol{k}}) \equiv \langle \phi_{\boldsymbol{k}'}|V|\psi_{\boldsymbol{k}}\rangle = g + g\Pi(2\varepsilon_{\boldsymbol{k}})T(\boldsymbol{k},\boldsymbol{k}';2\varepsilon_{\boldsymbol{k}})$$



Ultraviolet divergence

Retarded pair propagator shows an UV divergence

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$$\sum_{\boldsymbol{p}=\frac{2\pi}{L}(n_x,n_y,n_z)} f(\boldsymbol{p}) = \left(\frac{L}{2\pi}\right)^3 \int d^3\boldsymbol{p}\, f(\boldsymbol{p})$$

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*$L = 1$ is taken

$$= -\frac{m}{2\pi^2}\int_0^{\Lambda} dp \left[1 + \frac{k^2}{p^2 - k^2 - i\delta}\right] \xrightarrow{\Lambda \gg k} -\frac{m}{2\pi^2}\left[\Lambda + \int_0^{\infty} dp \frac{k^2}{p^2 - k^2 - i\delta}\right]$$

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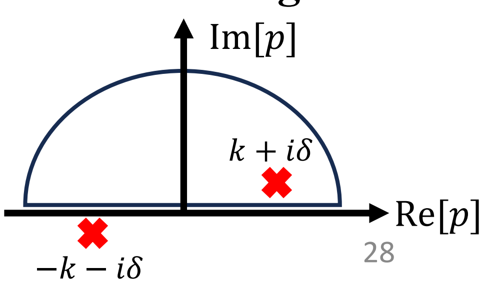
$$= -\frac{m}{2\pi^2}\left[\Lambda + \frac{k^2}{2}\int_{-\infty}^{\infty} dp \frac{1}{(p - k - i\delta)(p + k + i\delta)}\right]$$

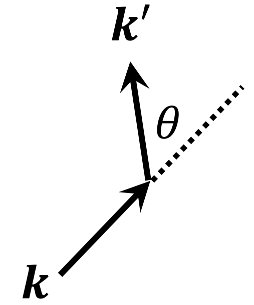
$$= -\frac{m}{2\pi^2}\left(\Lambda + \frac{\pi}{2} ik\right)$$

Large Λ cannot be taken, but T matrix is non-divergent

Contour integral



Partial wave expansion of the scattering amplitude



$$f(\boldsymbol{k}, \boldsymbol{k}') = \sum_{\ell=0} (2\ell + 1) f_\ell(\boldsymbol{k}, \boldsymbol{k}')\, P_\ell(\cos\theta)$$

ℓ: relative angular momentum of two particles

$P_\ell(\cos\theta)$: Legendre polynomial for the relative angle θ between $\boldsymbol{k}$ and $\boldsymbol{k}'$

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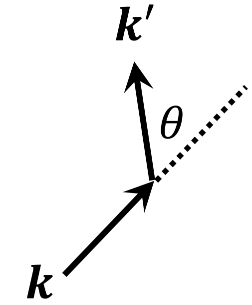

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S-wave Scattering amplitude

$$f_{\ell=0}(\boldsymbol{k},\boldsymbol{k}') = \frac{1}{k\cot\delta_k - ik}$$

$$\simeq \frac{1}{-1/a_s - ik}$$

Phase shift (effective range expansion)

$$k\cot\delta_k = -\frac{1}{a_s} + \frac{1}{2}k^2 r_{\mathrm{e}} \simeq -\frac{1}{a_s} \quad (r_{\mathrm{e}} \simeq 0)$$

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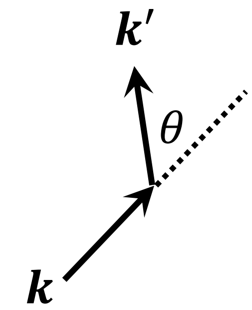

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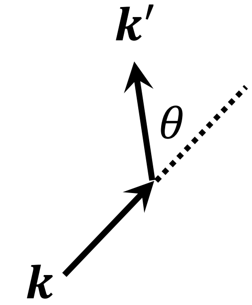

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Renormalization of g to a_s

$$\boxed{\frac{m}{4\pi a_s} = \frac{1}{g} + \frac{m\Lambda}{2\pi^2}}$$

Outline

- Theoretical model
- UV divergence and renormalization
- Physical interpretation of the scattering length
- Short summary

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Meaning of *S*-wave scattering length

Asymptotic wave function

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) \xrightarrow{r\to\infty} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \frac{f_{\ell=0}(\boldsymbol{k},\boldsymbol{k}')}{r} e^{ikr}$$

Low-momentum limit

$$\xrightarrow{k\to0} 1 - \frac{a_s}{r}$$

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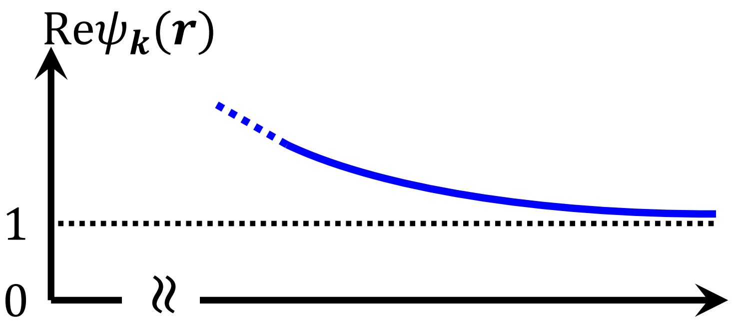

$\boldsymbol{a_s < 0}$: **weakly attractive**
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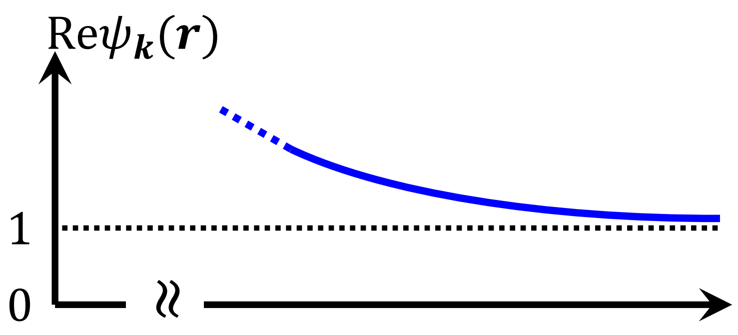

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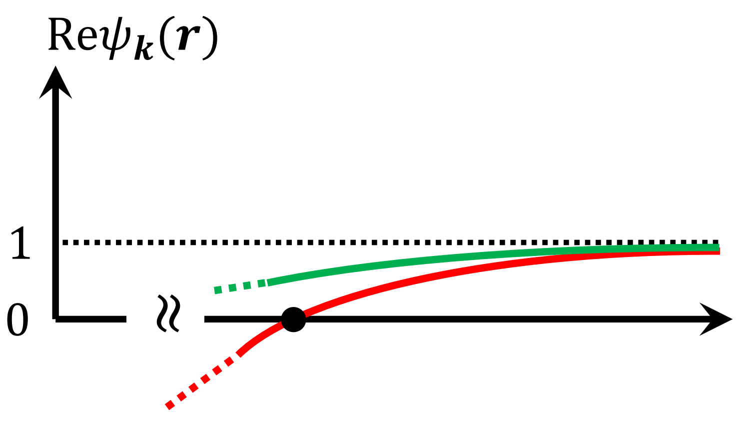

$\boldsymbol{a_s > 0}$: **strongly attractive**
(with a bound state, i.e., node ●)

weakly repulsive

One cannot distinguish two possibilities from the elastic scattering

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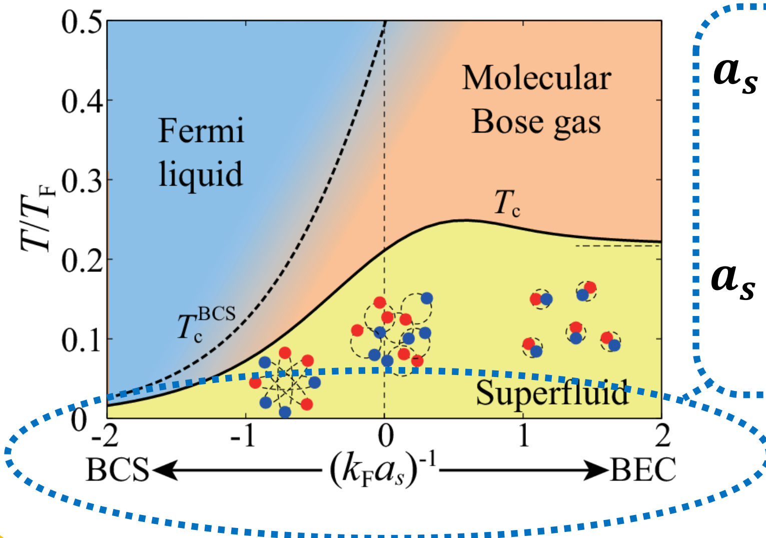
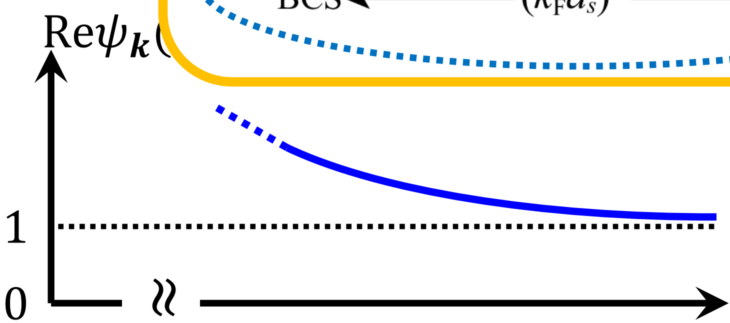


$\boldsymbol{a_s} < \mathbf{0}$: **weakly attractive**
➡ **BCS regime**

$\boldsymbol{a_s} > \mathbf{0}$: **strongly attractive**
➡ **BEC regime**

it



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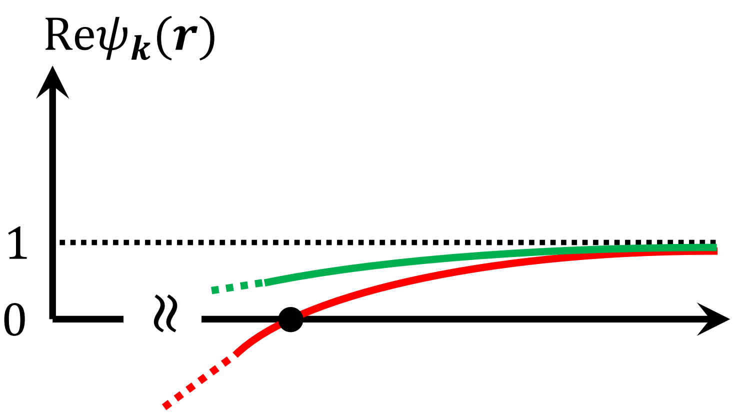

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One cannot distinguish two possibilities from the elastic scattering

Two-body problem

Let us check the relationship between the sign of a_s and the existence of bound state

Two-body Schrodinger equation: $\left[-\frac{\boldsymbol{\nabla}^2}{m} + g\delta(\boldsymbol{r})\right]\Psi_2(\boldsymbol{r}) = E\Psi_2(\boldsymbol{r})$

$\boldsymbol{r}$: relative distance between two particles

$\Psi_2(\boldsymbol{r})$: two-particle wave function

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$$\Rightarrow \sum_{\boldsymbol{k}} \left[\frac{k^2}{m} + g\delta(\boldsymbol{r}) - E\right]\chi_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 0$$

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$$\Rightarrow \sum_{\boldsymbol{k}}\left[\frac{k^2}{m}+g\delta(\boldsymbol{r})-E\right]\chi_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}=0 \quad \Rightarrow \left(\frac{p^2}{m}-E\right)\chi_{\boldsymbol{p}}=-g\sum_{\boldsymbol{k}}\chi_{\boldsymbol{k}}\equiv-g\Psi_2(\mathbf{0})$$

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$$\chi_{\boldsymbol{k}} = -\frac{g\Psi_2(\boldsymbol{0})}{k^2/m - E} \quad \Rightarrow \quad \boxed{\Psi_2(\boldsymbol{r}) = -g\Psi_2(\boldsymbol{0})\sum_{\boldsymbol{k}} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2/m - E}}$$

Two-body problem

Two-body wave function

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Two-body bound state = negative energy state $\boldsymbol{E = -E_{\mathrm{b}}}$

$E_{\mathrm{b}} > 0$: two-body binding energy

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$$1 = -\frac{mg}{(2\pi)^3}\int_0^\Lambda 4\pi k^2 dk \frac{1}{k^2 + mE_\mathrm{b}}$$ **: UV divergence!**

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Renormalization relation

$$\boxed{\frac{1}{g} = \frac{m}{4\pi a_s} - \frac{m\Lambda}{2\pi^2} \equiv \frac{m}{4\pi a_s} - \sum_{|\boldsymbol{k}|\leq\Lambda}\frac{m}{k^2}}$$

Regularized equation for E_{b}

$$-\frac{m}{4\pi a_s} = \sum_{\boldsymbol{k}}\left[\frac{m}{k^2 + mE_{\mathrm{b}}} - \frac{m}{k^2}\right]$$

Binding energy and scattering length

Regularized equation for $\boldsymbol{E}_{\mathbf{b}}$

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Binding energy and scattering length

Regularized equation for E_b

$E_b \neq 0$ only for $a_s > 0$

$$\boxed{E_\mathrm{b} = \frac{1}{m a_s^2}\theta(a_s)}$$

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$$\boxed{k = \sqrt{mE_\mathrm{b}}\tan\theta \qquad dk = \sqrt{mE_\mathrm{b}}\frac{d\theta}{\cos^2\theta}}$$

$$= \frac{m\sqrt{mE_\mathrm{b}}}{2\pi^2}\int_0^{\tan^{-1}\left(\frac{\Lambda}{\sqrt{mE_\mathrm{b}}}\right)} d\theta = \frac{m\sqrt{mE_\mathrm{b}}}{2\pi^2}\tan^{-1}\left(\frac{\Lambda}{\sqrt{mE_\mathrm{b}}}\right) \xrightarrow{\Lambda \gg \sqrt{mE_\mathrm{b}}} \frac{m\sqrt{mE_\mathrm{b}}}{4\pi}$$

Binding energy and scattering length

Regularized equation for E_b

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Binding energy and scattering length

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Critical coupling for bound state

$$E_{\mathrm{b}} \to 0 \text{ at } a_s \to \infty \quad \Rightarrow \quad g_{\mathrm{c}} = -\frac{2\pi^2}{m\Lambda}$$

In 3D, sufficiently large coupling is required for the formation of bound state

Binding energy and scattering length

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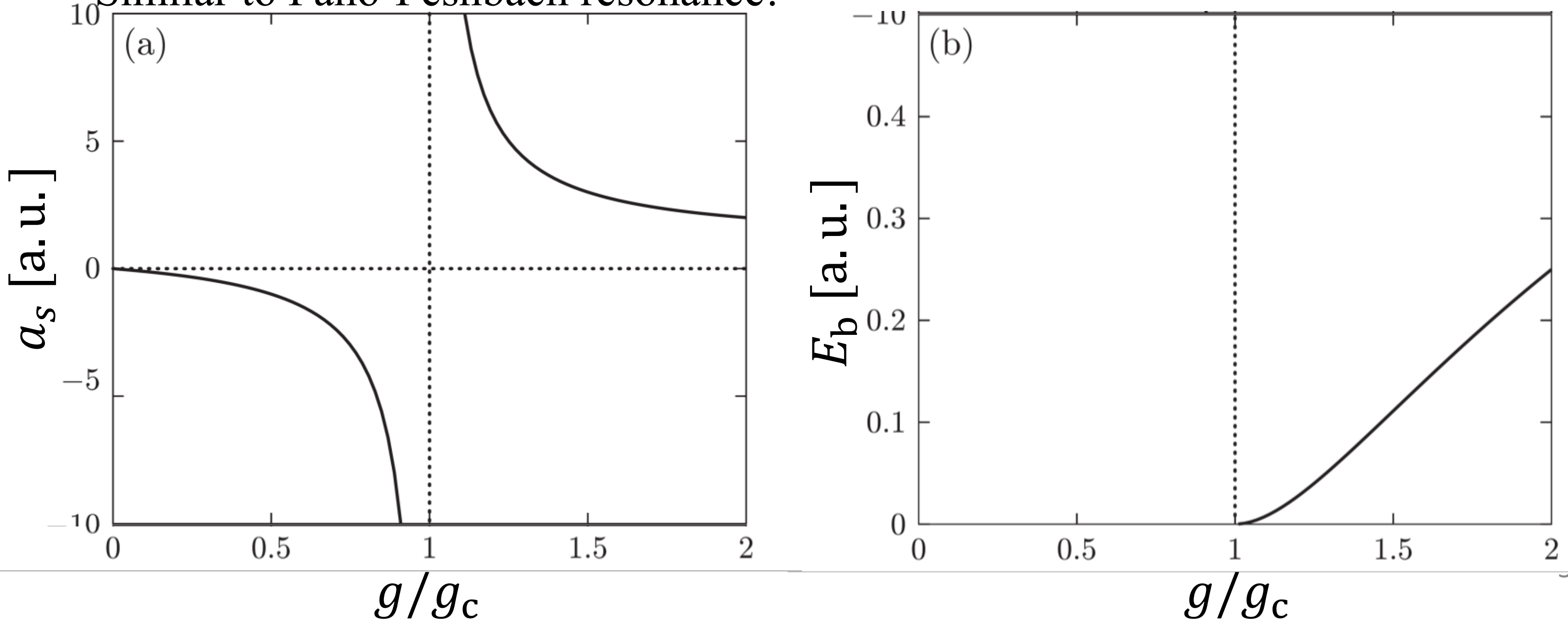

Binding energy and scattering length

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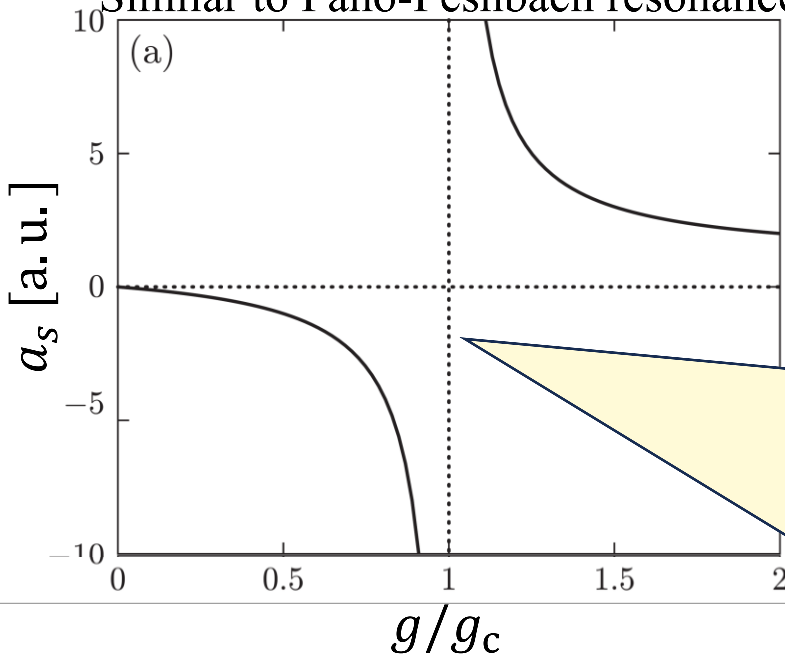

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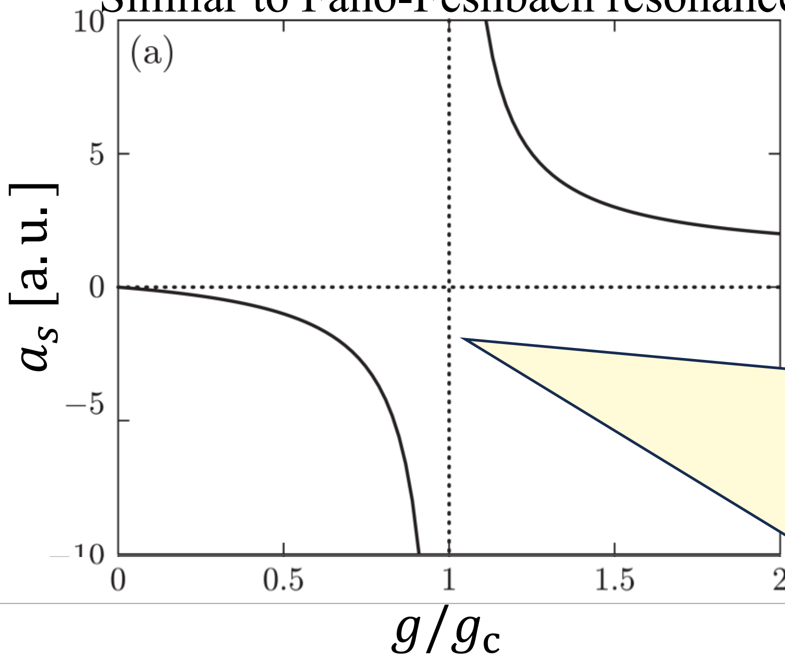

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Unitary Fermi gas

$$\boxed{E = \xi E_\mathrm{FG}}$$

E_FG: internal energy of free Fermi gas

EOS is independent of any parameter except for $\xi \simeq 0.37$ (exp.) : Bertsch parameter

Localized two-body wave function

Bound-state wave function

$$\boxed{\Psi_2(\boldsymbol{r}) = -g\Psi_2(\boldsymbol{0})\sum_{\boldsymbol{k}} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2/m + E_{\mathrm{b}}}}$$

$$\Psi_2(\boldsymbol{r}) = \frac{m|g|\Psi_2(\boldsymbol{0})}{4\pi^2}\int_0^{\pi} \sin\theta d\theta \int_0^{\infty} k^2 dk \frac{e^{ikr\cos\theta}}{k^2 + a_s^{-2}} \quad (E_{\mathrm{b}} = 1/ma_s^2)$$

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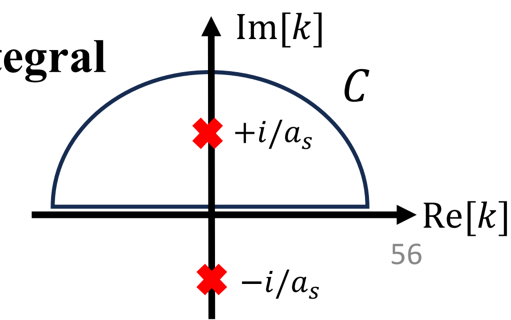
Contour integral



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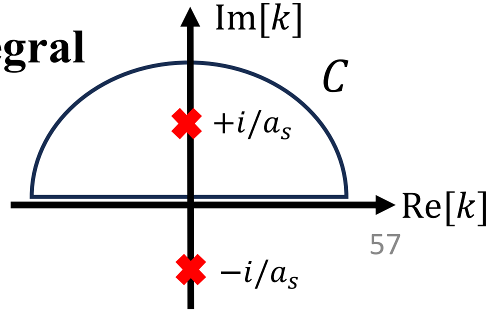

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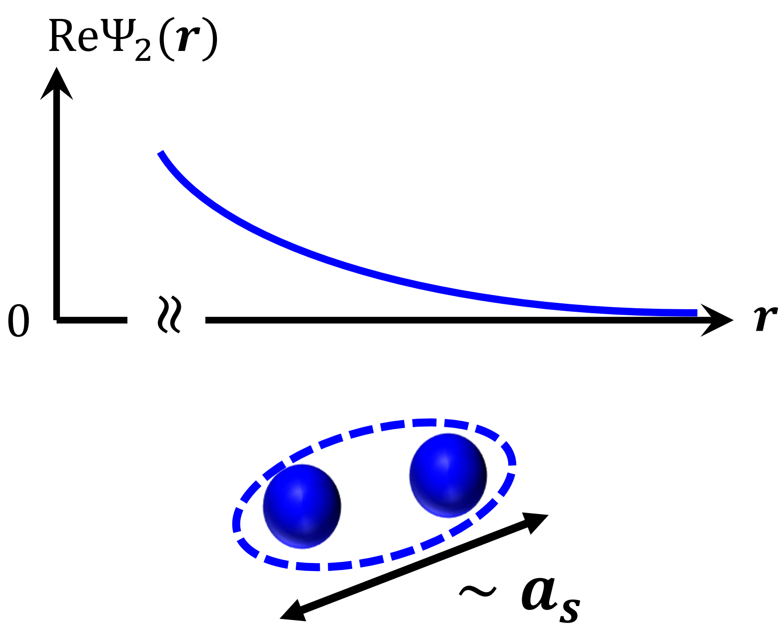
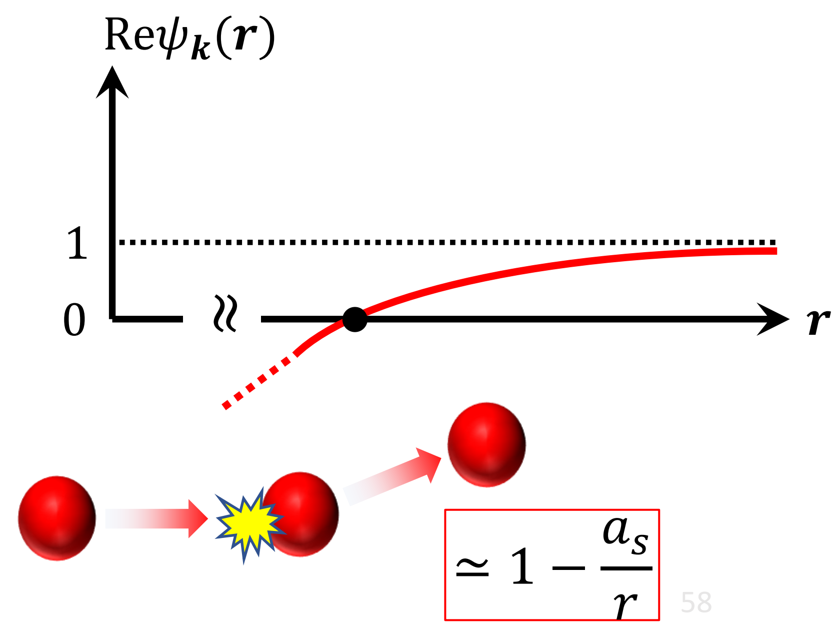
Bound state wave function (ground state) is localized with the size a_s
Note that it is different from the elastic scattering state (corresponding to excited state)

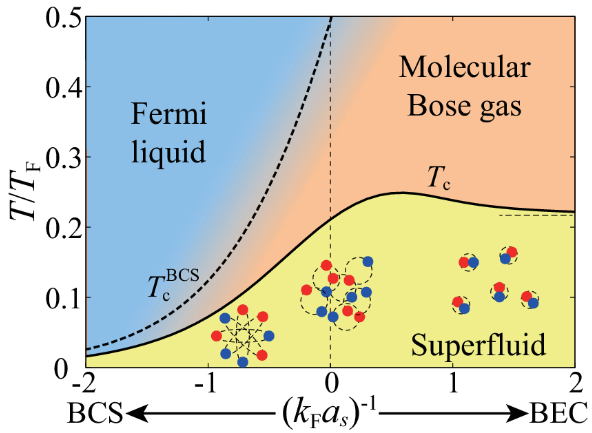
Bound state $(E = -E_b < 0)$



Scattering state $(E > 0)$



Why $1/(k_\mathrm{F} a_s)$?



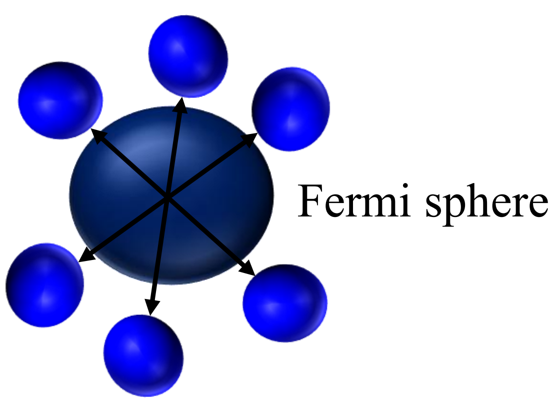
$k_\mathrm{F} = (3\pi^2\rho)^{1/3}$: **Fermi momentum for given $\boldsymbol{\rho}$**

$\rho = \sum_{|\boldsymbol{k}|\leq k_\mathrm{F}} 1$: **Particle number density**

➡ **mean interparticle distance $d \simeq \rho^{-1/3} \propto k_\mathrm{F}^{-1}$**

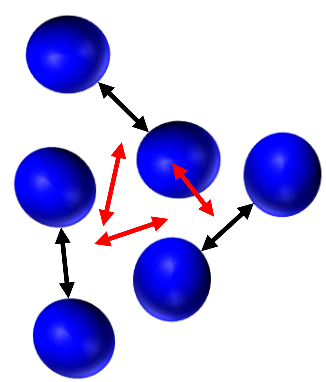
$\boxed{1/(k_\mathrm{F} a_s) < 0}$

No bound states (BCS Cooper pair)



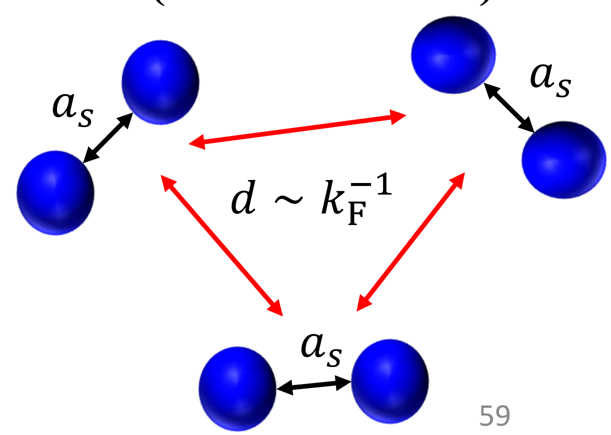
$\boxed{d/a_s \simeq 1/(k_\mathrm{F} a_s) \lesssim 1}$

Pairs are strongly overlapped (crossover regime)



$\boxed{d/a_s \simeq 1/(k_\mathrm{F} a_s) \gg 1}$

Pairs are point-like (molecular BEC)



Renormalization group perspective

Cutoff Λ and coupling constant g are not observable

➡ Λ is assumed to be an RG scale and g can be tuned such that actual observables are unchanged

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Dimensionless running coupling

$$u = \frac{m\Lambda}{2\pi^2}g$$

Dimensionless running scale

$$\Lambda \to \Lambda_s \equiv \Lambda e^{-s}$$

$$\frac{\partial u}{\partial s} = \frac{\partial \Lambda \mathrm{e}^{-s}}{\partial s}\left(\frac{m}{2\pi^2}g + \frac{m}{2\pi^2}\Lambda\frac{m}{2\pi^2}g^2\right)$$

$$\boxed{\frac{\partial u}{\partial s} = -u - u^2}$$

Renormalization group perspective

See also P. Nikolic and S. Sachdev, Phys. Rev. A **75**, 033608 (2007).

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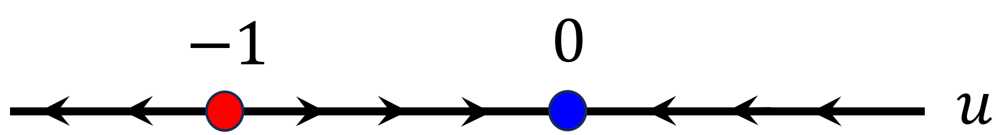

*Arrow indicates the flow direction with increasing s

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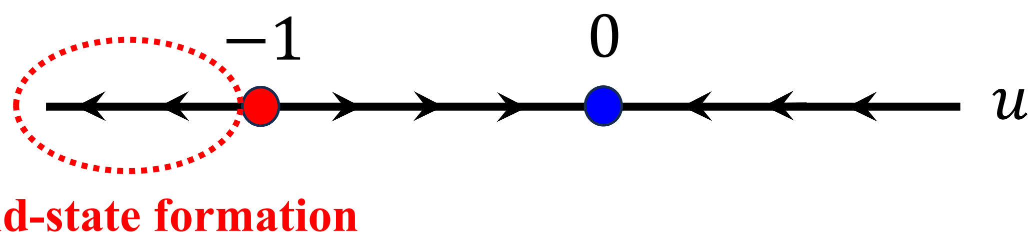

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● $u = -1$: UV unstable fixed point

$g_c = -\frac{2\pi^2}{m\Lambda}$: **critical coupling**

● $u = 0$: IR (infrared) stable fixed point

➡Free gas limit (no bound states)

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- Theoretical model
- UV divergence and renormalization
- Physical interpretation of the scattering length
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Summary of Part 2

- Contact-type interaction model is quite simple and relevant to describe physics in ultracold Fermi gases.
- However, we have encountered an UV divergence.
- Accordingly, the coupling constant should be renormalized such that experimental observables (e.g., scattering amplitude) are reproduced. The interaction strength is then characterized by the scattering length.