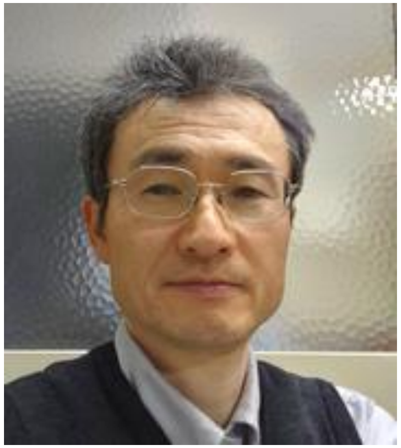
hadron-quark crossover:
Insights from ultracold atomic physics

Hiroyuki Tajima

The University of Tokyo, Japan

References: HT, K. Iida, T. Kojo, and H. Liang, PRL **135**, 042701 (2025).

Collaborators



Kei Iida
The Open University of Japan

Toru Kojo
KEK

Haozhao Liang
The University of Tokyo

Outline

- **Introduction**

  Can we study a microscopic mechanism of hadron-quark crossover in cold atom physics?

- **Formulation**

  Tripling fluctuation theory

- **Results**

  Equation of state and momentum distributions

- **Summary**

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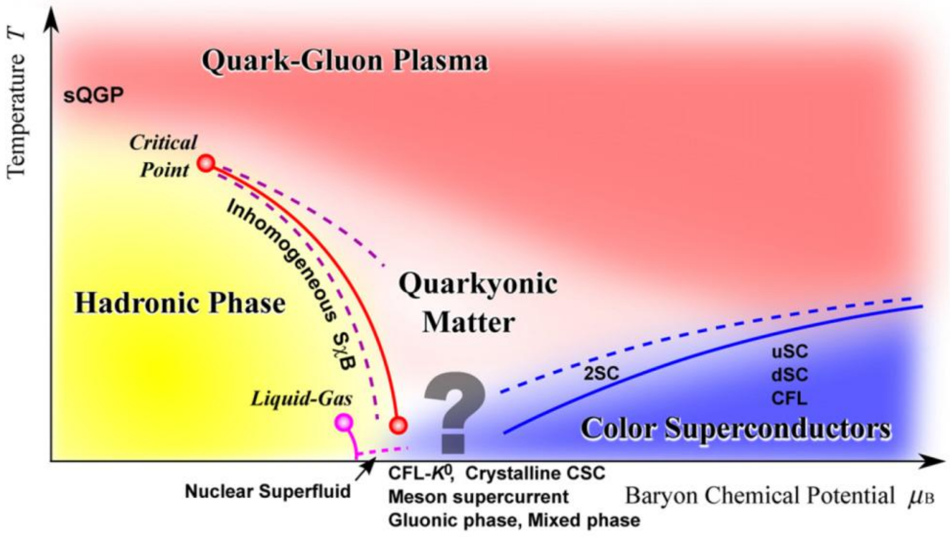
Extremely dense matter

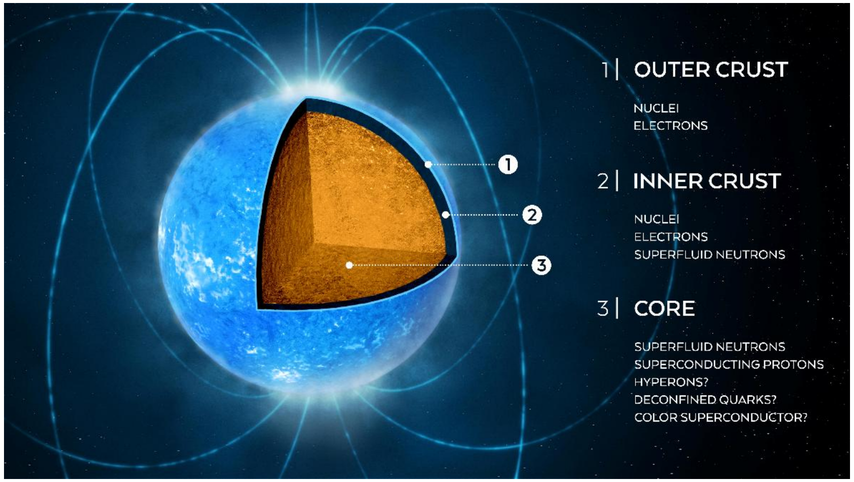
How does the hadronic phase change into quark matter at finite densities?

Dense QCD phase diagram



K. Fukushima, *et al*., Rep. Prog. Phys. **74**, 014001 (2011).

Neutron star as a testing ground of dense matter



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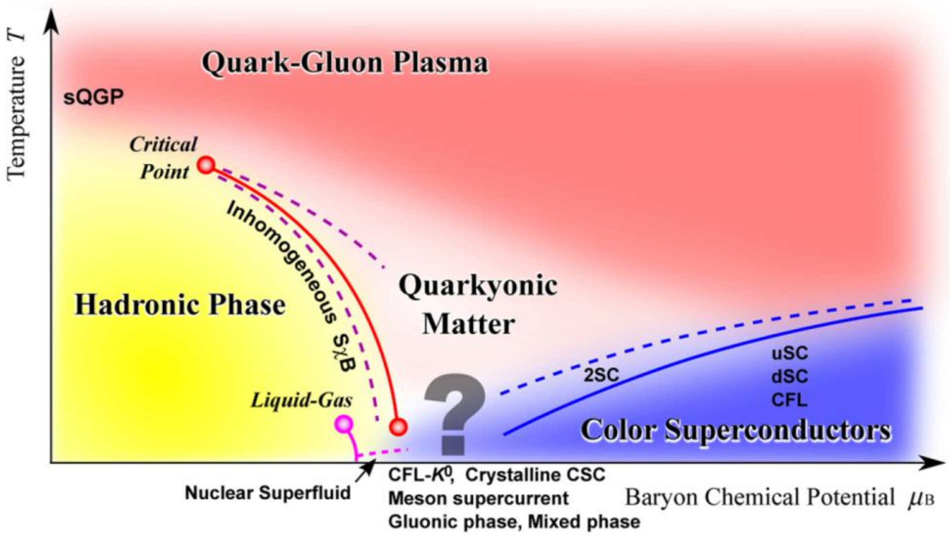

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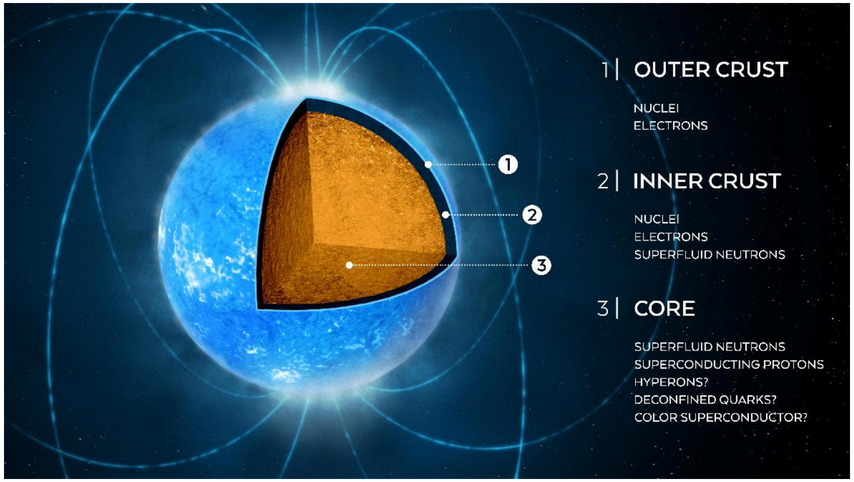

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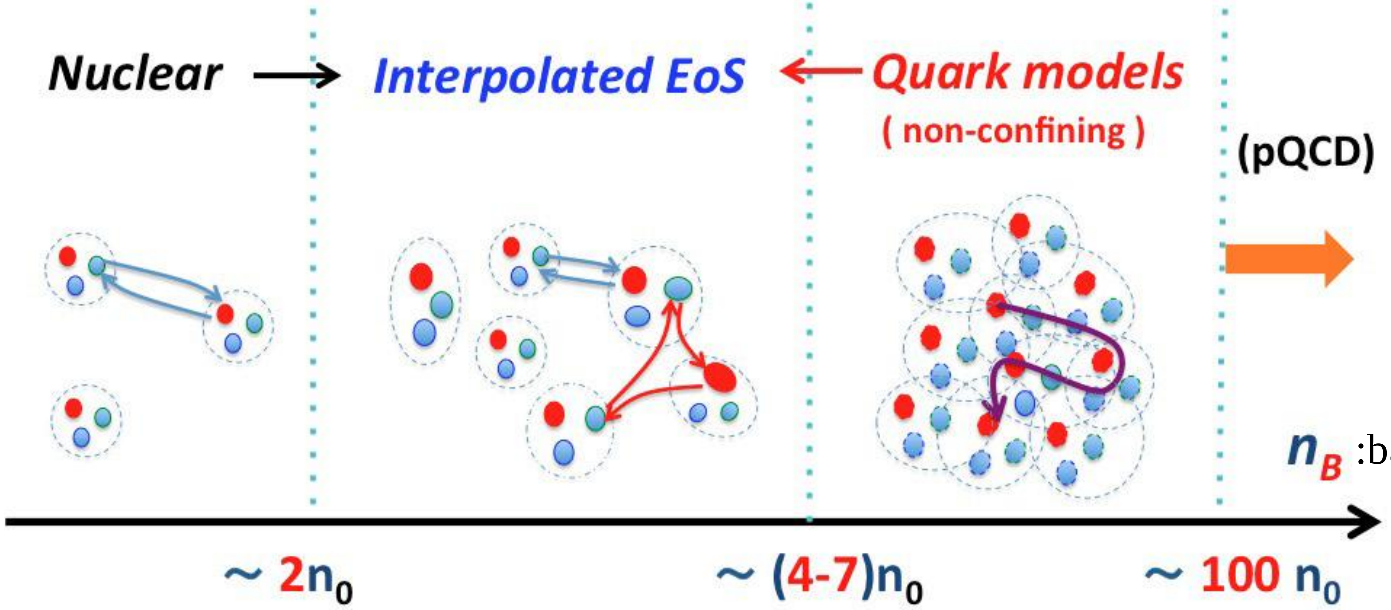
× **Sign problem in lattice QCD**

× **Limited information in the observation**

Hadron-quark (HQ) crossover

G. Baym, *et al.*, Rep. Prog. Phys. **81**, 056902 (2018).



:baryon density

normal nuclear density

$n_0 = 0.16\ \mathrm{fm}^{-3}$

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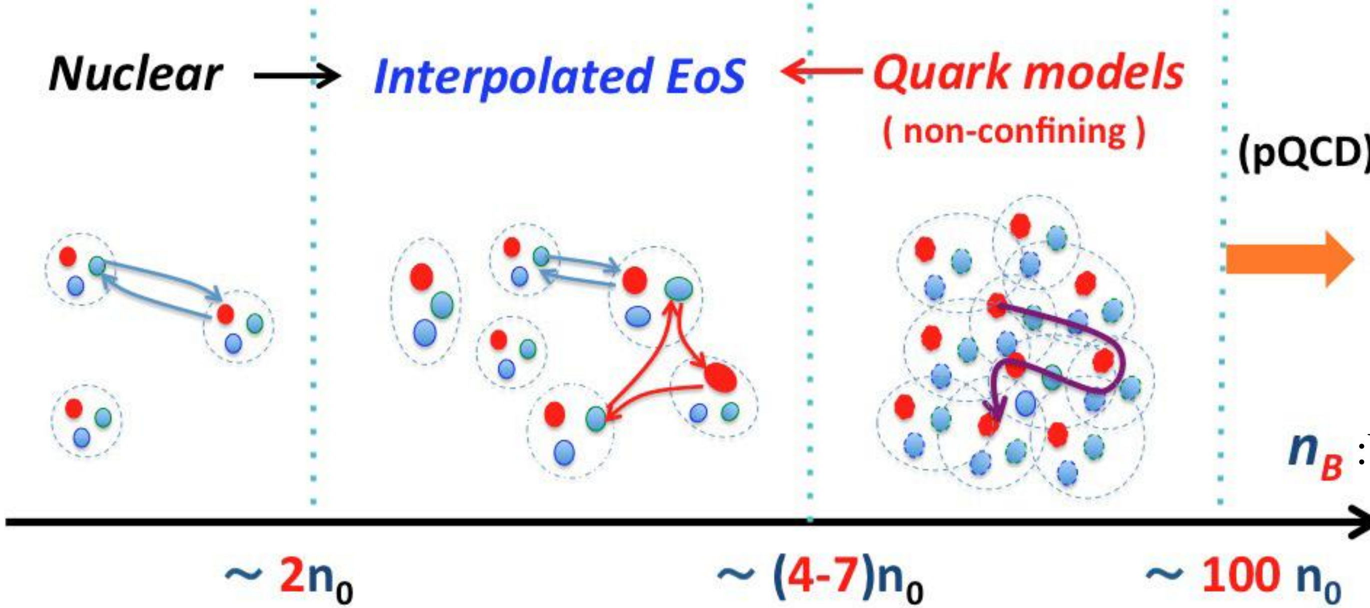
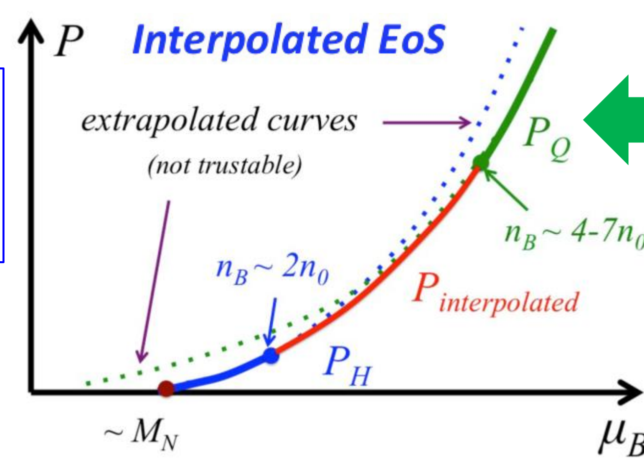


:baryon density

Ab initio simulation with realistic nuclear force, Chiral EFT, etc…



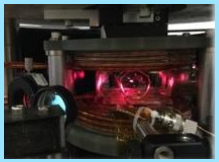
Quark matter EOS (e.g., NJL with mean field)

normal nuclear density

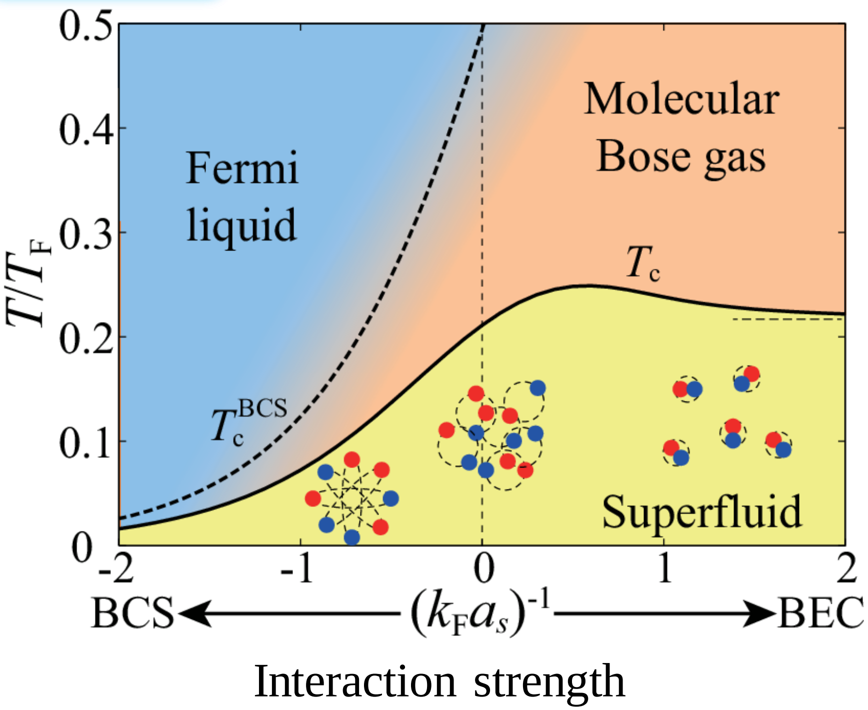
$n_0 = 0.16\ \mathrm{fm}^{-3}$

Analogy with BEC-BCS crossover?

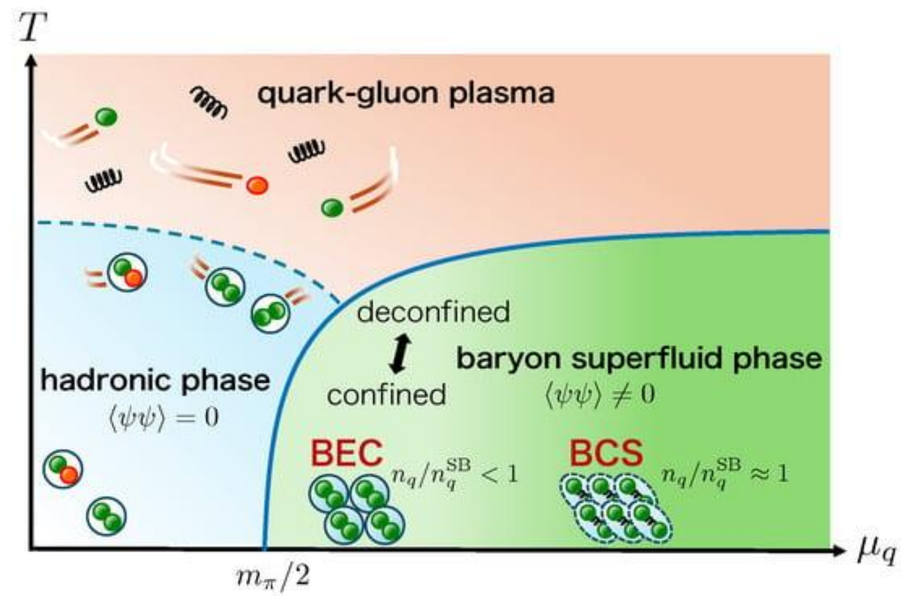
Review: Y. Ohashi, HT, and P. van Wyk, Prog. Part. Nucl. Phys. **111**, 103739 (2020).



BEC-BCS crossover realized in ultracold Fermi gases



≃ HQ crossover in "2-color" QCD

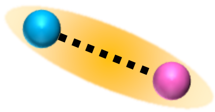


D. Suenaga, Symmetry **17**, 124 (2025).

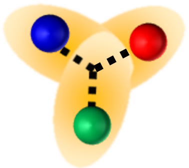
Photo: http://www.sci.osaka-cu.ac.jp/phys/laser/research_Li.html

BEC-BCS crossover
≃ HQ crossover in “3-color” QCD?

Dimer
“boson”



Baryon
“fermion”

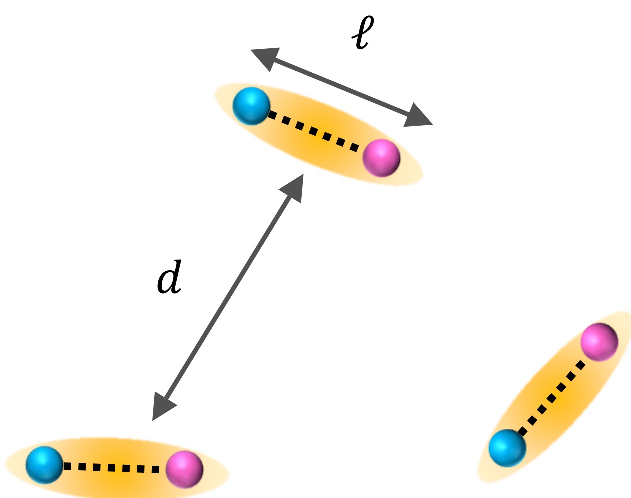


BEC-BCS crossover $\simeq$ HQ crossover in "3-color" QCD?

Let us consider density evolution

Dimer BEC

$(\ell \ll d)$

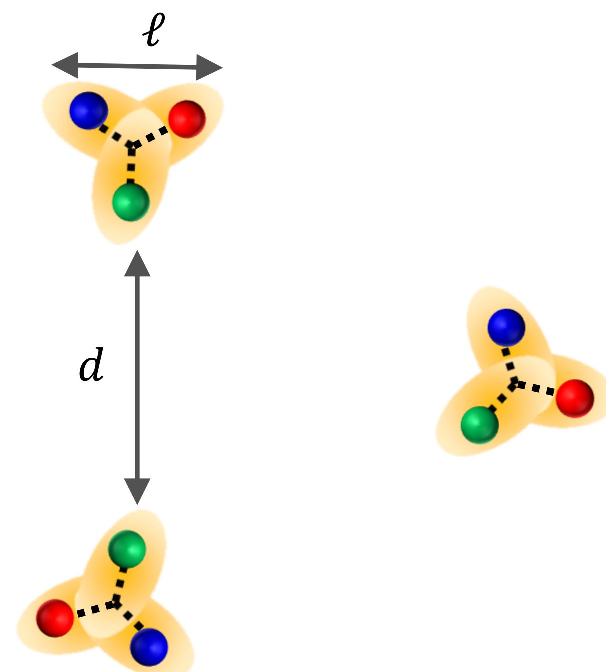


d: interparticle distance

ℓ: molecular size

Dilute nuclear matter

$(\ell \ll d) \quad \Leftrightarrow n_B \ll n_0$



*$d \sim 0.75$ fm at $n = n_0 = 0.16$ fm^{-3}

*Proton size ~0.9 fm

BEC-BCS crossover
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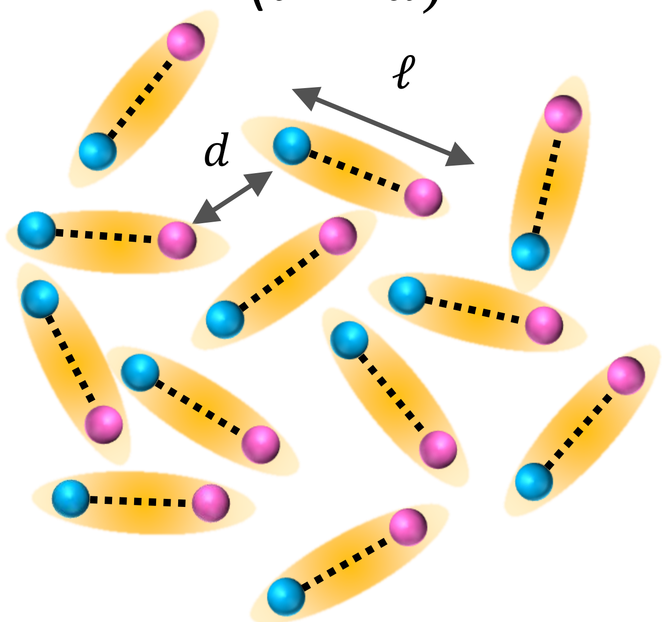
Let us consider density evolution

Dense dimer gas

$(\ell \simeq d)$



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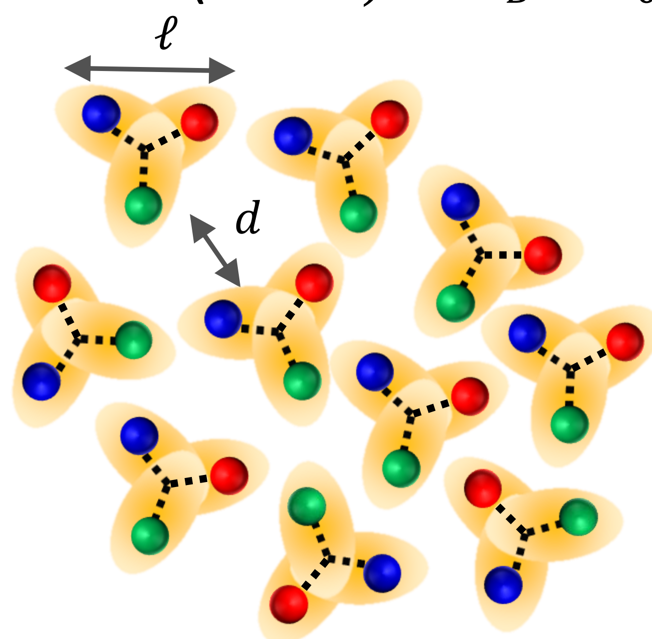

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BEC-BCS crossover
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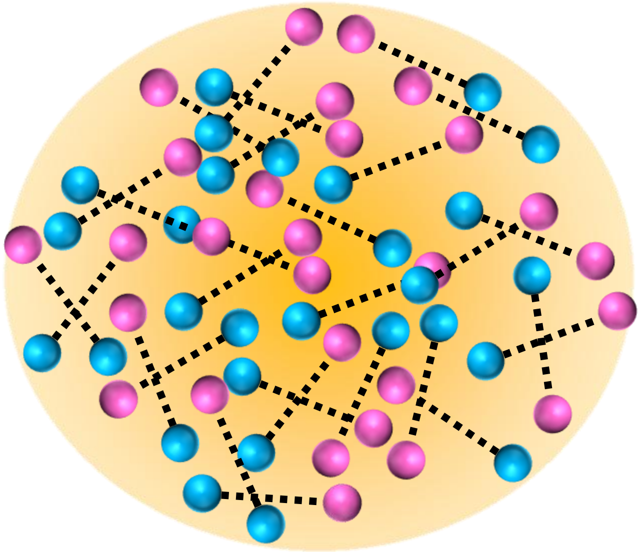
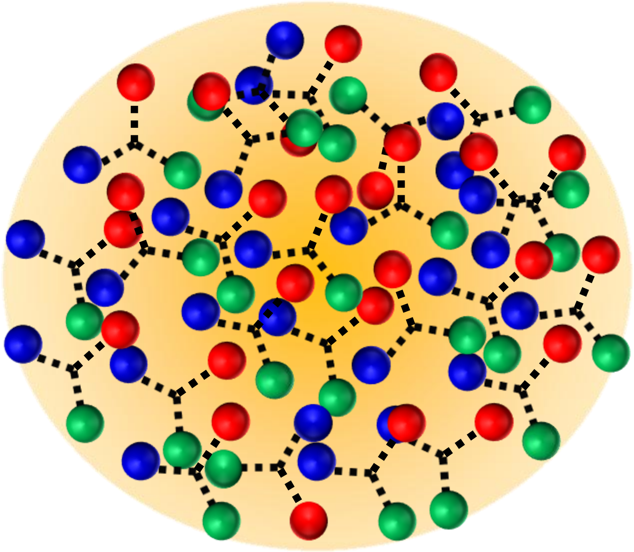
Let us consider density evolution

Crossover to the BCS phase
($d < \ell$)



Crossover to quark matter
($d < \ell$) ⇔ $n_B > n_0$



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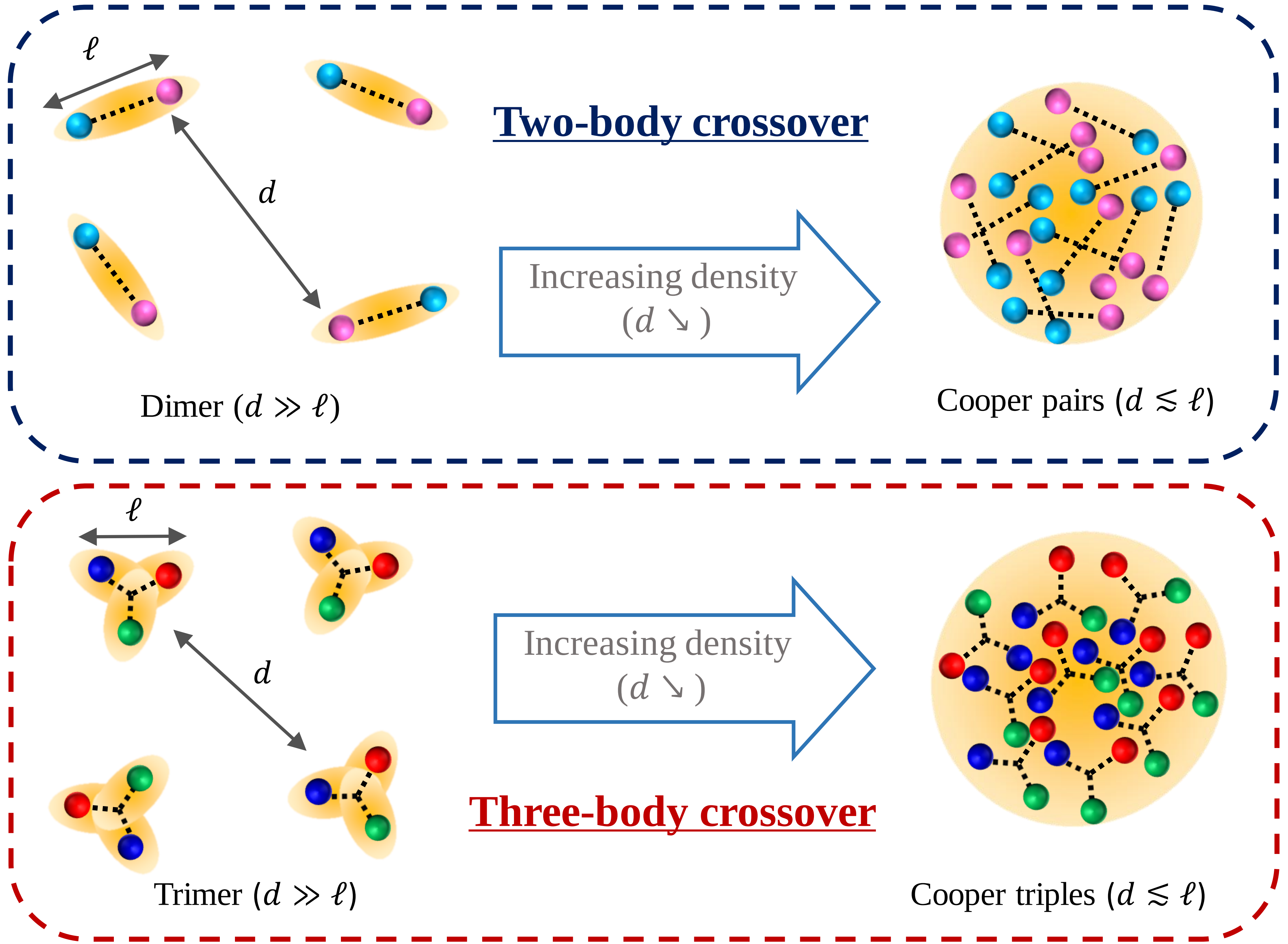
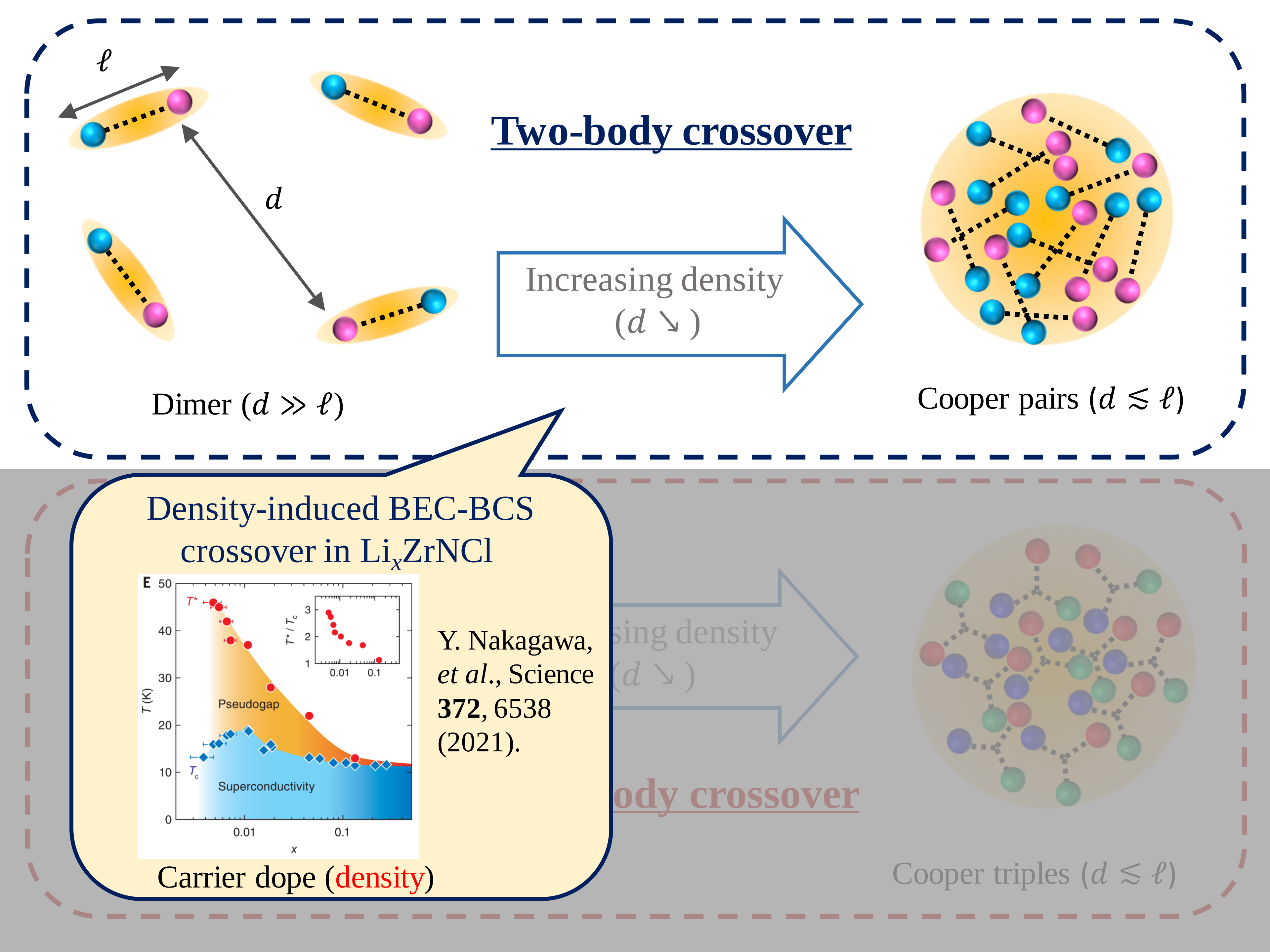
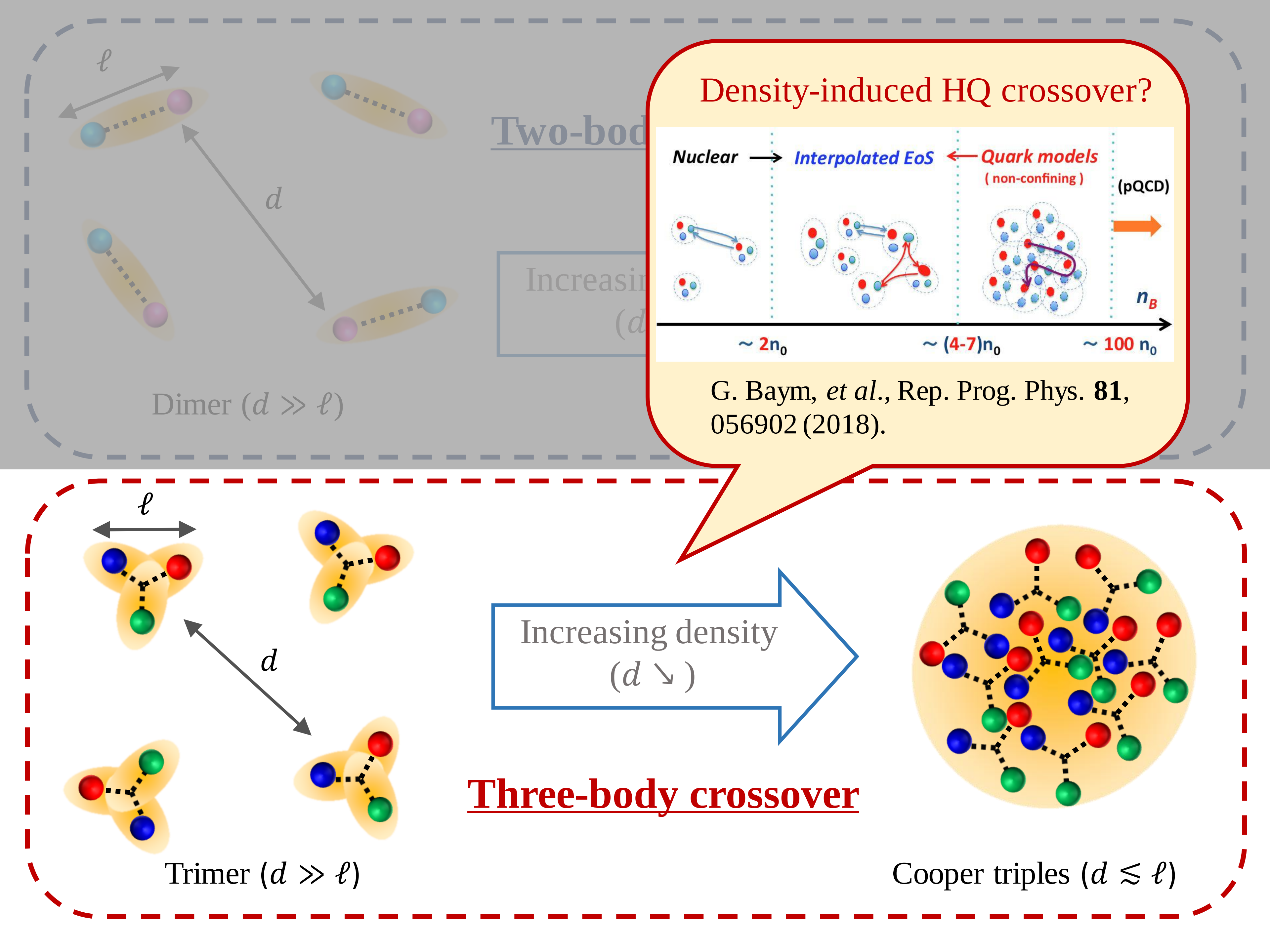


Two-body crossover
ℓ
d
Increasing density
($d \searrow$)
Dimer ($d \gg \ell$)
Cooper pairs ($d \lesssim \ell$)
Three-body crossover
ℓ
d
Increasing density
($d \searrow$)
Trimer ($d \gg \ell$)
Cooper triples ($d \lesssim \ell$)



ℓ
d
Two-body crossover
Increasing density
(d ↘)
Dimer (d ≫ ℓ)
Cooper pairs (d ≲ ℓ)
Density-induced BEC-BCS crossover in Li_xZrNCl
E
T*
Tc
T (K)
Pseudogap
Superconductivity
T* / Tc
0.01
0.1
x
Y. Nakagawa, et al., Science 372, 6538 (2021).
Carrier dope (density)
sing density
(d ↘)
ody crossover
Cooper triples (d ≲ ℓ)



ℓ
d
Two-bod
Increasi
(d
Dimer (d ≫ ℓ)
Density-induced HQ crossover?
Nuclear
Interpolated EoS
Quark models
(non-confining)
(pQCD)
n_B
~ $2n_0$
~ $(4\text{-}7)n_0$
~ $100\ n_0$
G. Baym, *et al.*, Rep. Prog. Phys. **81**, 056902 (2018).
ℓ
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Increasing density
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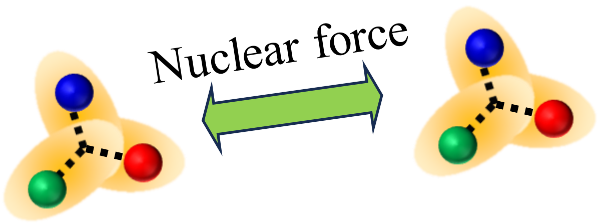
Theoretical approaches to nuclear equation of state (EOS)

Conventional nuclear EOS
= Effective theory of nucleons
(Nucleons + nuclear force)

$$H = H_0 + V_{NN} + V_{NNN} + \cdots$$



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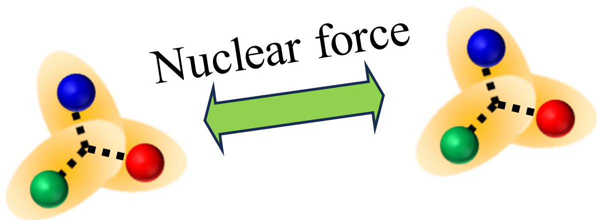

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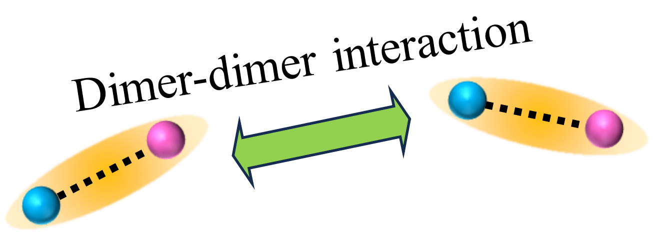
Effective theory of dimers
➡ never describe the crossover regime



Molecular BEC EOS

$$E = \frac{N}{2}E_b + N\frac{\pi\hbar^2 a_{dd}}{2m} \times n\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{na_{dd}^3} + \ldots\right)$$

N. Navon et al., Science **328**, 729 (2010).

a_{dd}: dimer-dimer scattering length

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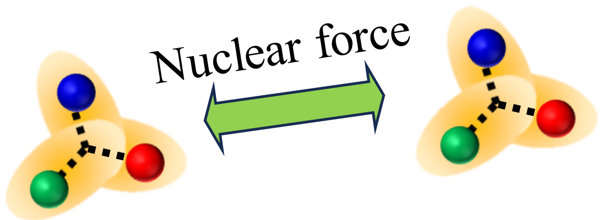
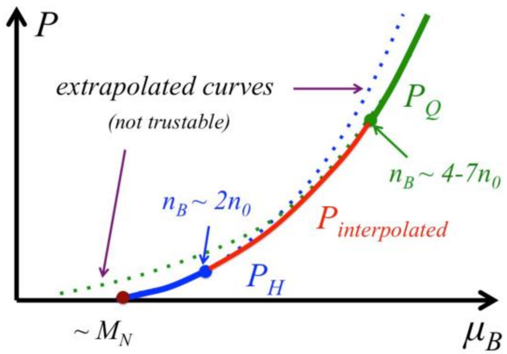


Interpolated EOS based on the hadron-quark crossover



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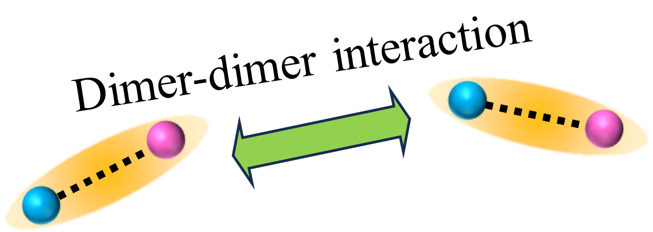

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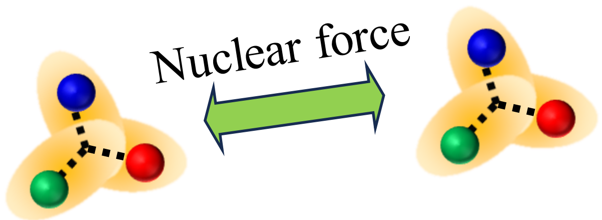

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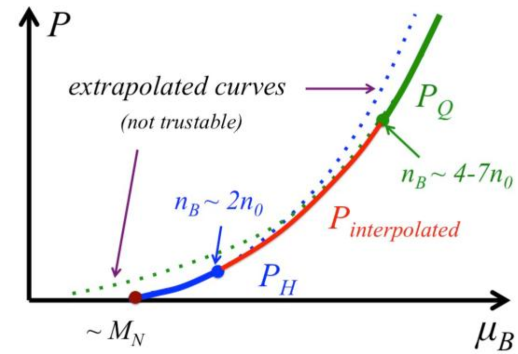

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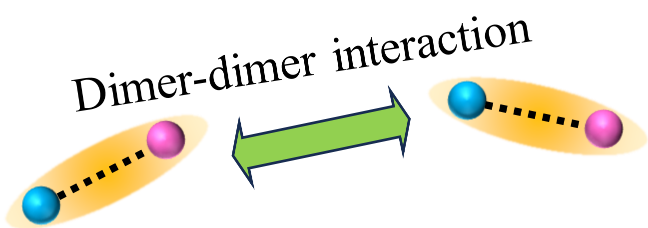

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N. Navon et al., Science **328**, 729 (2010).

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Phenomenologically interpolating BCS and BEC EOS

➡ **no microscopic foundation**

Molecular BEC EOS

$$E = \frac{N}{2}E_b + N\frac{\pi\hbar^2 a_{dd}}{2m} \times n\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{na_{dd}^3} + \ldots\right)$$

Interpolate!

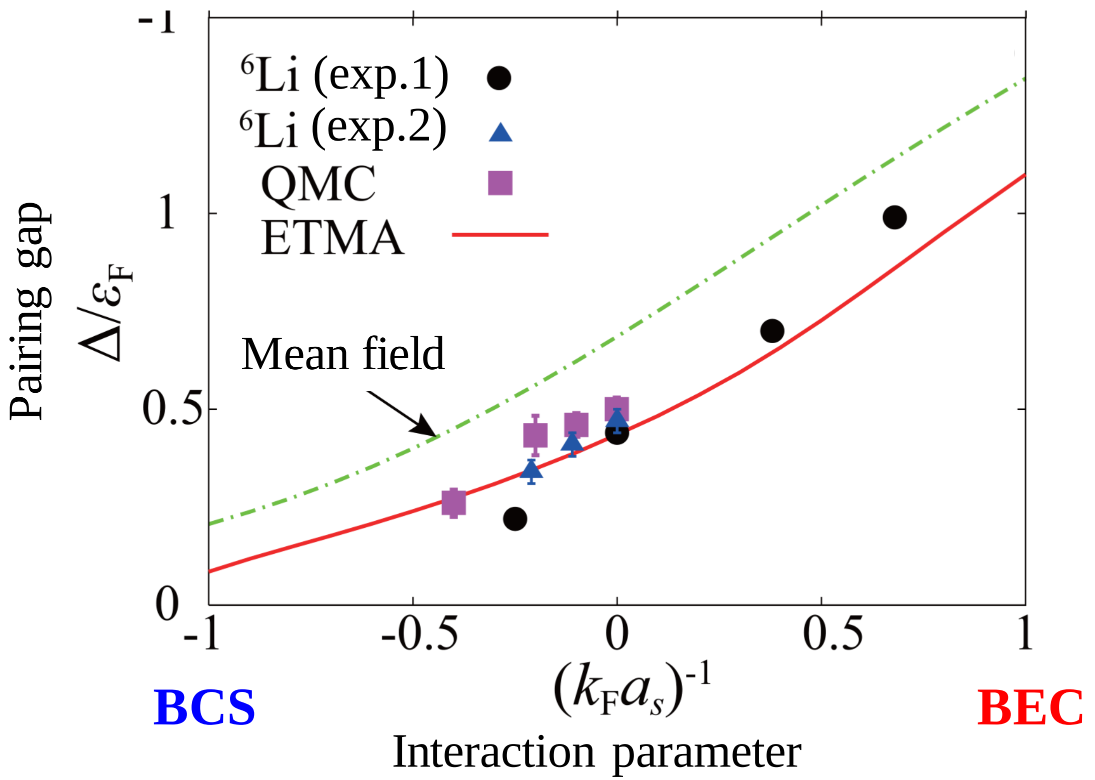
Fermi gas EOS at weal coupling

$$E = \frac{3}{5}NE_F\left(1 + \frac{10}{9\pi}k_F a + 0.18(2)(k_F a)^2 + 0.03(2)(k_F a)^3 + \ldots\right)$$

Many-body theory for the crossover

In the case of the BEC-BCS crossover, the mean-field (BCS-Eagles-Leggett) theory is “qualitatively” valid at zero temperatures.

Mean field = Superfluid/superconducting order parameter Δ

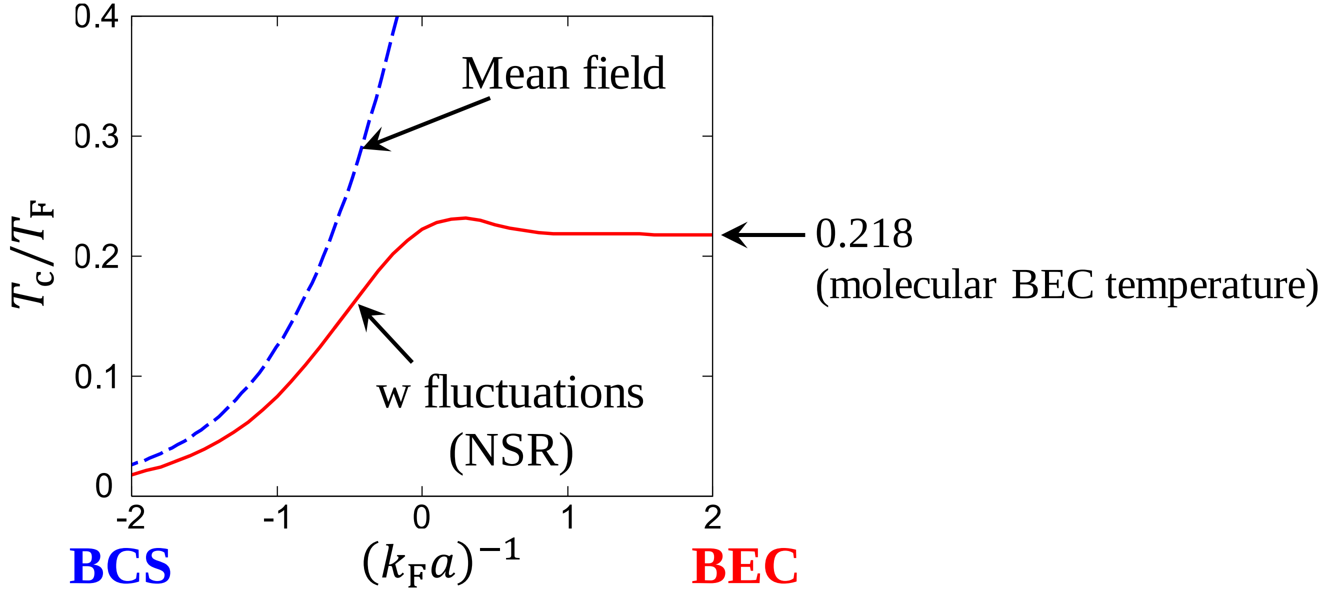


Y. Ohashi, HT, and P. van Wyk, Prog. Part. Nucl. Phys. **111**, 103739 (2020).

Many-body theory for the crossover

In the absence of order parameters for crossover (e.g., hadron-quark crossover), the mean-field theory is **INVALID** even qualitatively

Critical temperature:
$\Delta(T = T_c) = 0$



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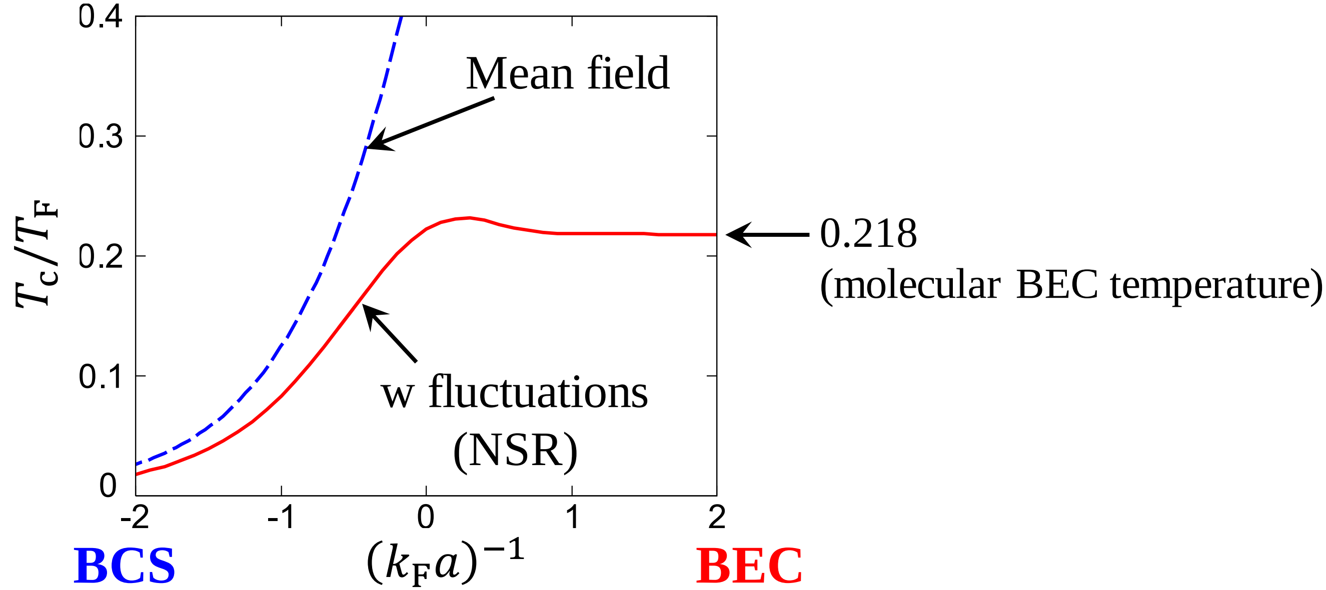
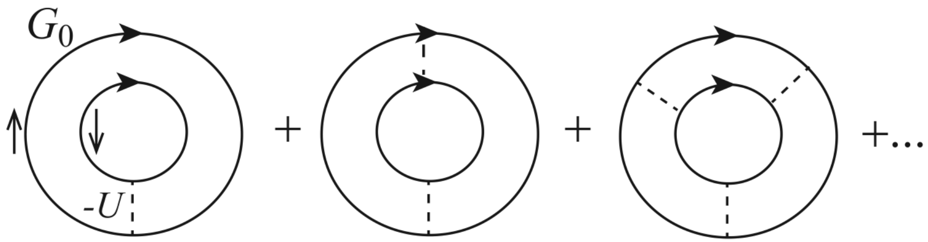


Nozieres-Schmitt-Rink (NSR) approach to pairing fluctuations

$\Omega_{NSR} =$

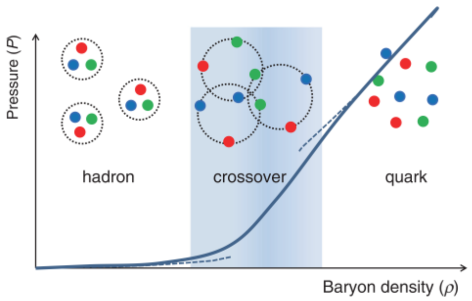


P. Nozières, and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).

➡ **Theory for "tripling" fluctuations is needed**

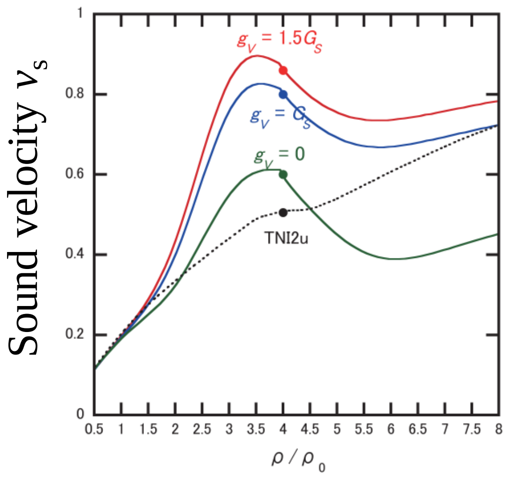
Two key points to understand the hadron-quark crossover

1. Peaked speed of sound



Rapid increase of $P(\rho)$

➡ $v_s^2 = \frac{dP}{dE}$ **exhibits a peak**



K. Masuda, T. Hatsuda, and T. Takatsuka, PTEP **2013,** 073D01 (2013).

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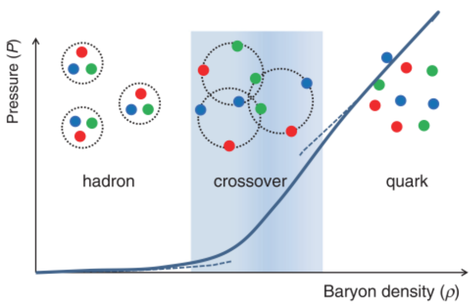

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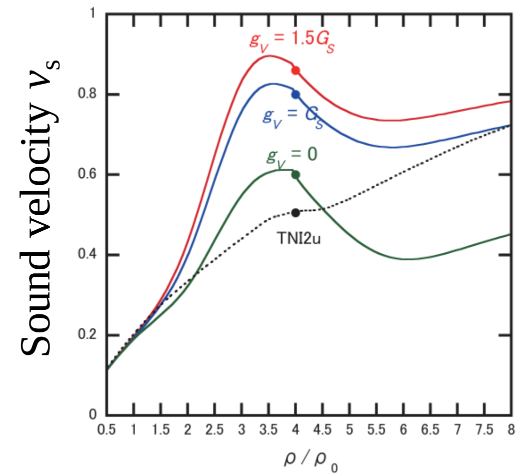

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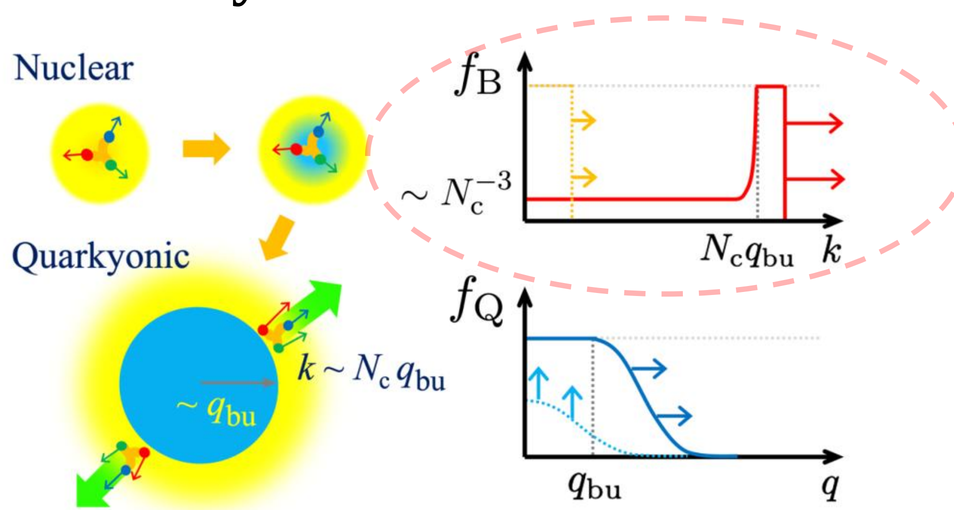
2. Non-monotonic behavior of baryon-momentum distribution

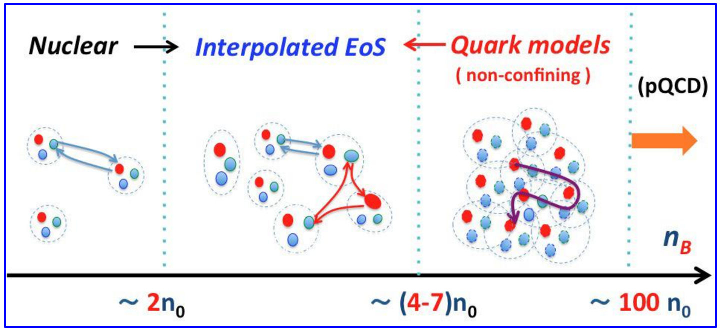
Explicit duality model:

Y. Fujimoto, T. Kojo, and L. D. McLerran, Phys. Rev. Lett. **132**, 112701 (2024).



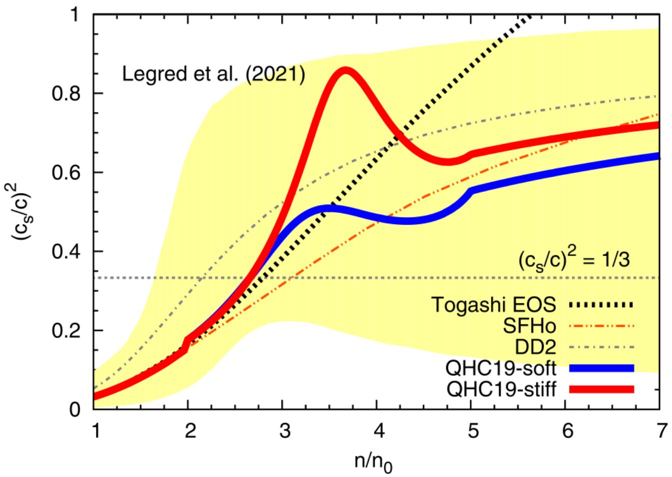
In this talk…

- In analogy with the BEC-BCS crossover, we discuss the role of tripling fluctuations in the hadron-quark crossover.
- Considering tripling fluctuations within the phase-shift representation of three-body propagators, we investigate equation of state as well as momentum distributions of fermions and three-body bound states.



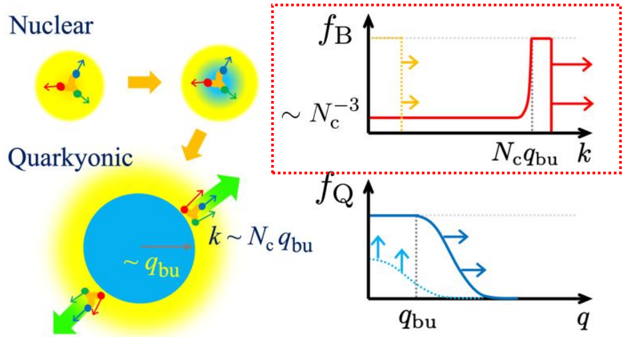
G. Baym, *et al.*, Rep. Prog. Phys. **81**, 056902 (2018).

Peaked speed of sound



Y.-J. Huang, *et al.*, Phys. Rev. Lett. **129**, 181101 (2022)

Momentum shell structure of baryons



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  Can we study a microscopic mechanism of hadron-quark crossover in cold atom physics?

- **Formulation**

  Tripling fluctuation theory

- **Results**

  Equation of state and momentum distributions

- **Summary**

N-body clustering fluctuations

R. Dashen, S.-K. Ma, and H. J. Bernstein, Phys. Rev. **187**, 345 (1969).

Clustering fluctuations on thermodynamic potential:

$$\delta\Omega_N = T\sum_{\boldsymbol{K}}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\ln\left(1+e^{-\omega/T}\right)\partial_\omega\varphi$$

***N*-body propagator and phase shift:** $\mathcal{G}/\mathcal{G}_0 \ = \ |\mathcal{G}/\mathcal{G}_0|e^{i\varphi}$

Exact constraint: $\varphi(\boldsymbol{K},\omega\to-\infty)=\varphi(\boldsymbol{K},\omega\to\infty)=0$

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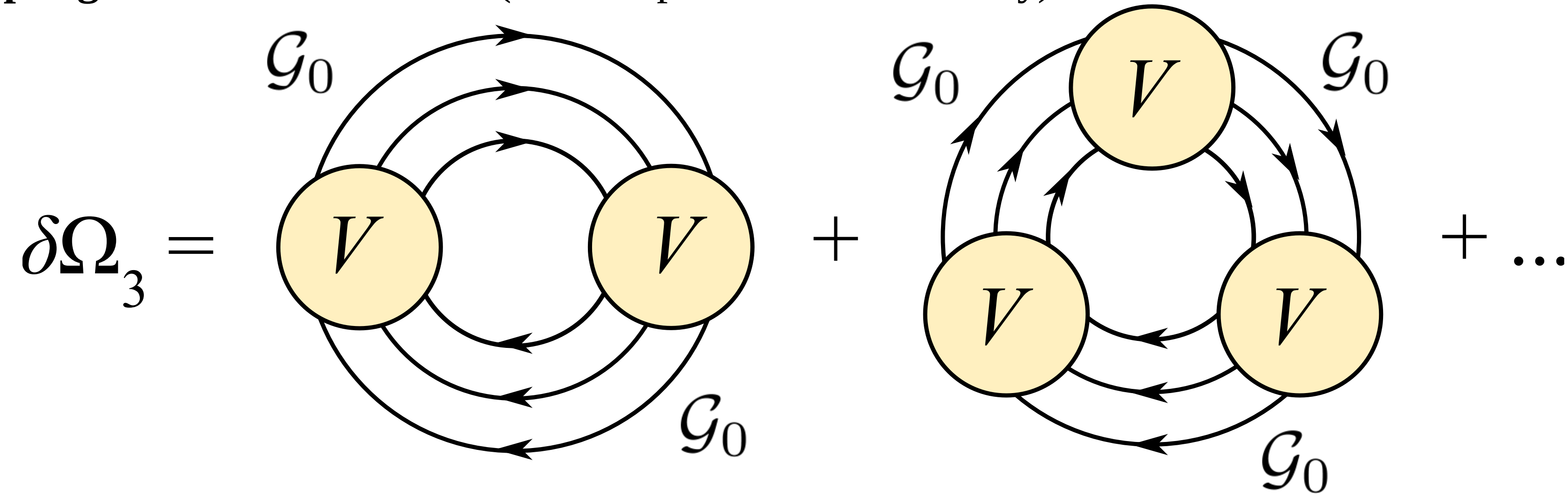
Exact constraint: $\varphi(\boldsymbol{K},\omega\to-\infty)=\varphi(\boldsymbol{K},\omega\to\infty)=0$

Tripling fluctuations: *N* = 3 (*N* = 2 reproduces NSR theory)



V: short-range interaction responsible for *N*-body cluster formation

Role of the phase shift

$$\delta\Omega_{N=3} = T\sum_{\boldsymbol{K}}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\ln\left(1+e^{-\omega/T}\right)\partial_{\omega}\,\varphi(\boldsymbol{K},\omega)$$

$$= \sum_{\boldsymbol{K}}\int_{-\infty}^{\infty}\frac{d\widetilde{\omega}}{\pi}\frac{1}{e^{(\omega+E_{\mathrm{B}}^{\mathrm{kin}}(K)-\tilde{\mu}_{\mathrm{B}})/T}+1}\varphi(\widetilde{\omega})$$

Integration by part & Changing variable

subtracted three-body energy: $\omega \to \widetilde{\omega} = \omega - E_{\mathrm{B}}^{\mathrm{kin}}(K) + \tilde{\mu}_{\mathrm{B}}$

Baryon kinetic energy:

$$E_{\mathrm{B}}^{\mathrm{kin}}(K) = K^2/2M_{\mathrm{B}}$$

Baryon chemical potential:

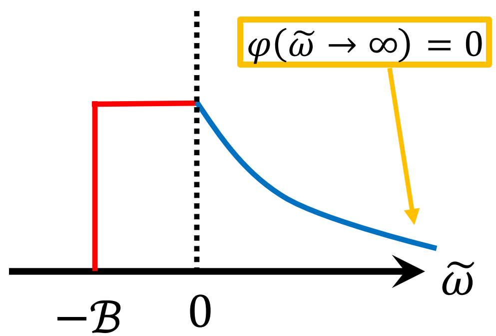
$$\tilde{\mu}_{\mathrm{B}} = 3\tilde{\mu} \equiv 3(\mu - \Sigma_{\mathrm{HF}})$$

Structure of the phase shift

Bound state **Scattering state**

$$\varphi(\widetilde{\omega}) = \pi\Theta(\widetilde{\omega}+\mathcal{B})\Theta(-\widetilde{\omega}) + \Theta(\widetilde{\omega})\varphi_{\mathrm{scatt.}}(\widetilde{\omega})$$

$\mathcal{B}$:Cluster binding energy



Bound state v.s. Scattering state

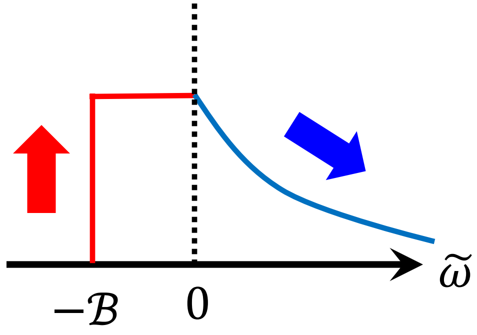
Tripling fluctuation contribution to the particle number density

$$\delta n = -\frac{\partial \delta \Omega_3}{\partial \mu} = 3 \sum_{\boldsymbol{K}} \int_{-\infty}^{\infty} d\tilde{\omega} \frac{1}{e^{(\omega + E_{\mathrm{B}}^{\mathrm{kin}}(K) - \tilde{\mu}_{\mathrm{B}})/T} + 1} \frac{\partial_{\tilde{\omega}} \varphi(\tilde{\omega})}{\pi}$$

Bound state v.s. **Scattering state**

$$\frac{\partial_{\tilde{\omega}} \varphi(\tilde{\omega})}{\pi} = \delta(\tilde{\omega} + \mathcal{B}) + \Theta(\tilde{\omega}) \frac{\partial_{\tilde{\omega}} \varphi_{\mathrm{scatt.}}(\tilde{\omega})}{\pi}$$

$> \mathbf{0}$ $< \mathbf{0}$



Bound state v.s. Scattering state

Tripling fluctuation contribution to the particle number density

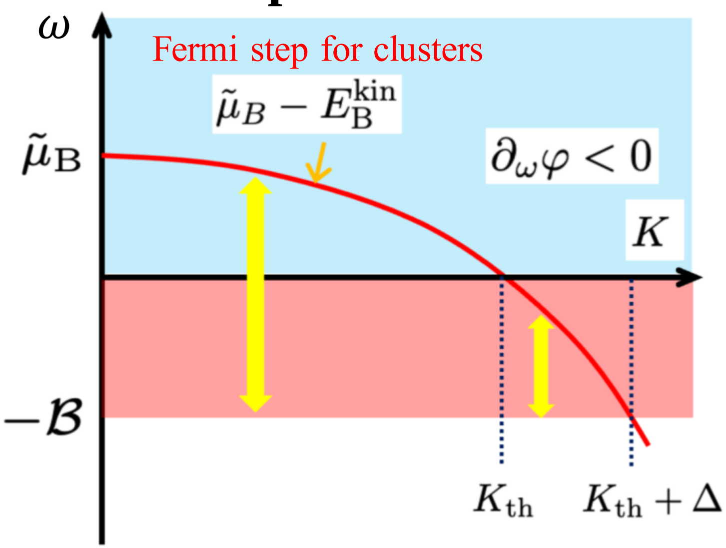
$$\delta n = -\frac{\partial \delta \Omega_3}{\partial \mu} = 3 \sum_{\boldsymbol{K}} \int_{-\infty}^{\infty} d\tilde{\omega} \frac{1}{e^{(\omega + E_{\mathrm{B}}^{\mathrm{kin}}(K) - \tilde{\mu}_{\mathrm{B}})/T} + 1} \frac{\partial_{\tilde{\omega}} \varphi(\tilde{\omega})}{\pi}$$

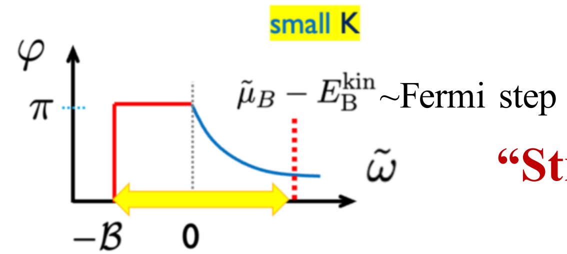
(Fermi step for clusters)

Bound state v.s. **Scattering state**

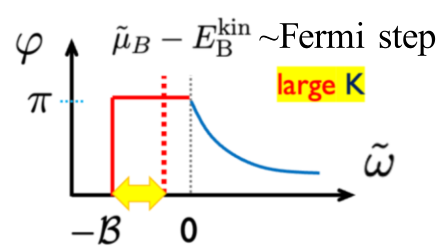
$$\frac{\partial_{\tilde{\omega}} \varphi(\tilde{\omega})}{\pi} = \delta(\tilde{\omega} + \mathcal{B}) + \Theta(\tilde{\omega}) \frac{\partial_{\tilde{\omega}} \varphi_{\mathrm{scatt.}}(\tilde{\omega})}{\pi}$$

$K - \omega$ plane





"Strong cancellation"



"No cancellation"

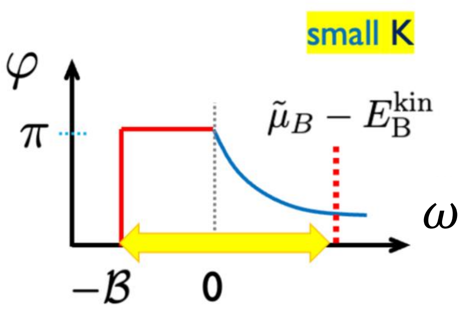
Baryon momentum distribution

Number density: $n = -\frac{\partial \Omega_{\mathrm{HF}}}{\partial \mu} - \frac{\partial \delta\Omega_3}{\partial \mu} \equiv 3\sum_{\boldsymbol{k}} f_Q(\boldsymbol{k}) + 3\sum_{\boldsymbol{K}} f_B(\boldsymbol{K})$

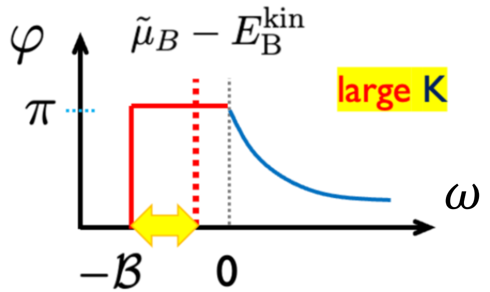
$$f_B(\boldsymbol{K}) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \partial_\omega \varphi(\omega) f(\omega - \tilde{\mu}_B + E_B^{\mathrm{kin}}(K))$$

Strong cancellation

No cancellation



Non-trivial CoM momentum (***K***) dependence arises via interplay of bound and scattering states



Difference btw pairing and tripling fluctuations

P. Nozieres and S. Schmitt-Rink, JLTP **59**, 195 (1985).

$$\Omega - \Omega_f = -\frac{1}{\pi}\sum_{\mathbf{q}} \int_{-\infty}^{+\infty} d\omega\, g(\omega)\delta(q,\omega)$$

$$\frac{1}{2}(N - N_f) = -\frac{\partial}{\partial\mu}(\Omega - \Omega_f) = \sum_{\mathbf{q}} \int_{-\infty}^{+\infty} \frac{d\omega}{\pi}\, g(\omega)\, \frac{\partial}{\partial\mu}\delta(q,\omega)$$

Pairing fluctuation case

$$g(\omega) = \frac{1}{e^{\beta\omega} - 1}$$

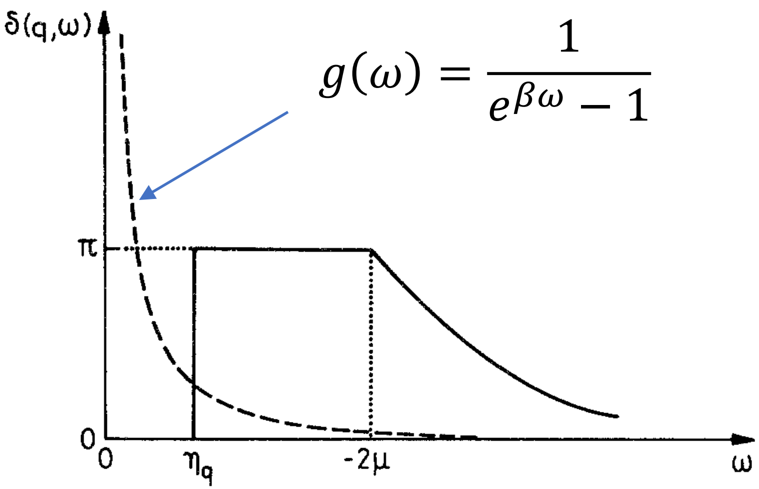


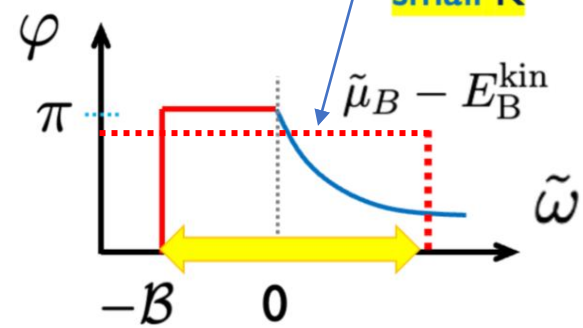
Fig. 7. The phase shift $\delta(q,\omega)$ in the dilute strong coupling limit. η_q corresponds to the bound state, -2μ to the continuum threshold. The dashed curve is the Bose factor $g(\omega)$.

$$\eta_q = \frac{q^2}{4m} - \mathfrak{B} \qquad \delta_b(q,\omega) = \pi\theta(\omega - \eta_q)$$

Tripling fluctuation case

$$f(\omega - \tilde{\mu}_B + E_B^{\rm kin}(K)) = \frac{1}{e^{\beta(\omega - \tilde{\mu}_B + E_B^{\rm kin}(K))} + 1}$$

small K



Even for pairing fluctuations, the cancellation can occur but masked by enhanced Bose distribution.

Outline

- **Introduction**

  Can we study a microscopic mechanism of hadron-quark crossover in cold atom physics?

- **Formulation**

  Tripling fluctuation theory

- **Results**

  Equation of state and momentum distributions

- **Summary**

How to demonstrate the crossover physics?
-1D nonrelativistic three-color fermions-

Latter section: 1D nonrelativistic (1DNR) three-color Fermi gases with three-body attraction

Why?

- Sign problem free Quantum Monte Carlo
- Similarity with HQ crossover
- Possible realization in future atomic experiments

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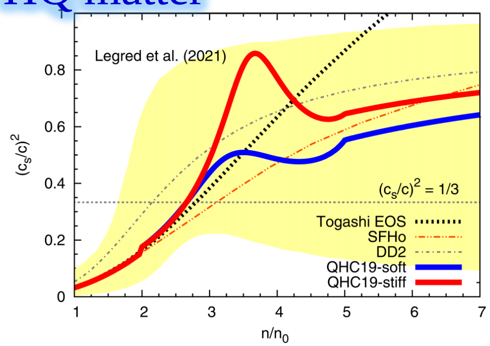
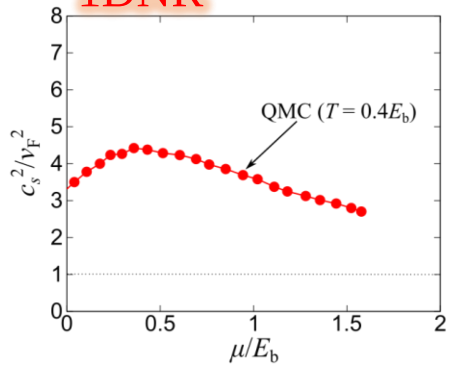
Peaked speed of sound

HQ matter



Y.-J. Huang, *et al.*, Phys. Rev. Lett. **129**, 181101 (2022)

1DNR



J. McKenny, *et al.*, Phys. Rev. A **102**, 023313 (2020).

Both systems exhibit a characteristic peaked behavior in the crossover regime

How to demonstrate the crossover physics?
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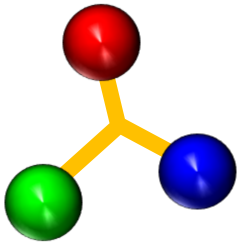
Asymptotic freedom and trace anomaly

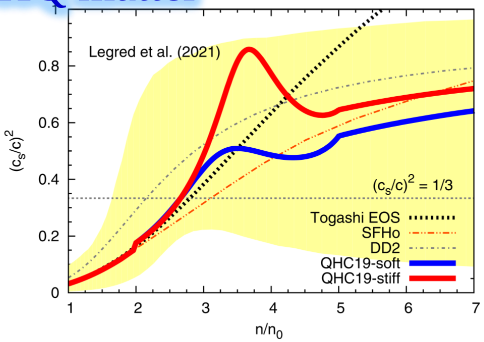


$$\frac{\partial g_3}{\partial \ln\Lambda} = \frac{m}{\sqrt{3\pi}} g_3^2$$

J. Drut, et al., Phys. Rev. Lett. **120**, 243002 (2018).

Peaked speed of sound

HQ matter



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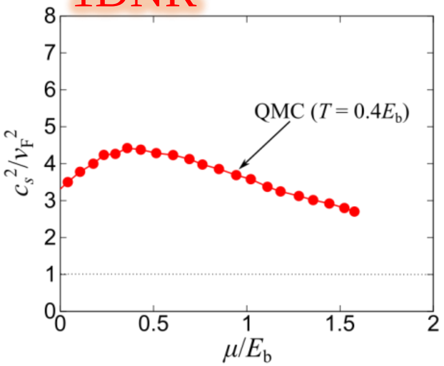
1DNR



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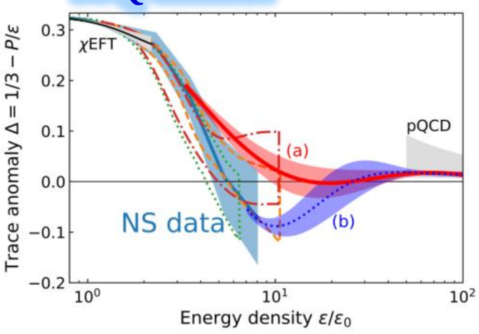
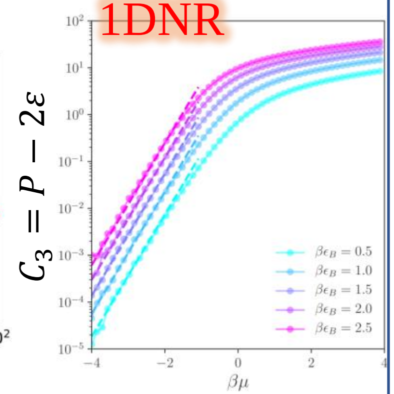


Y. Fujimoto, *et al.*, Phys. Rev. Lett. **129**, 252702 (2022).

1DNR



J. McKenny, *et al.*, Phys. Rev. A **102**, 023313 (2020).

Trace anomaly would influence EOS

How to demonstrate the crossover physics?
-1D nonrelativistic three-color fermions-

- Hamiltonian density: $\hat{H} = \hat{H}_0 + \hat{V}_3$

One-body kinetic term

$$\hat{H}_0 = \sum_{a=\mathrm{r,g,b}} \psi_a^\dagger \left(-\frac{\partial_x^2}{2m} - \mu \right) \psi_a$$

m: mass
μ: chemical potential
$a = \mathrm{r, g, b}$: pseudo-color (hyperfine states)
$\psi_a^\dagger, \psi_a$: fermionic field operator

Three-body interaction (involving quantum anomaly with asymptotic freedom)

J. Drut, et al., PRL **120**, 243002 (2018).

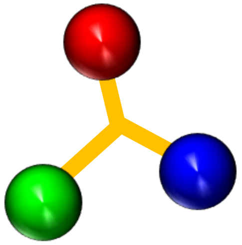
$$\hat{V}_3 = V(\psi_\mathrm{r}^\dagger \psi_\mathrm{r})(\psi_\mathrm{g}^\dagger \psi_\mathrm{g})(\psi_\mathrm{b}^\dagger \psi_\mathrm{b})$$

$V < 0$: three-body attraction

Three-body binding energy

$$\mathcal{B} = \frac{\Lambda^2}{m} \exp\left(\frac{2\sqrt{3}\pi}{mV} \right)$$

Λ: UV cutoff scale



Phase shift of three-body propagator

Three-body propagator

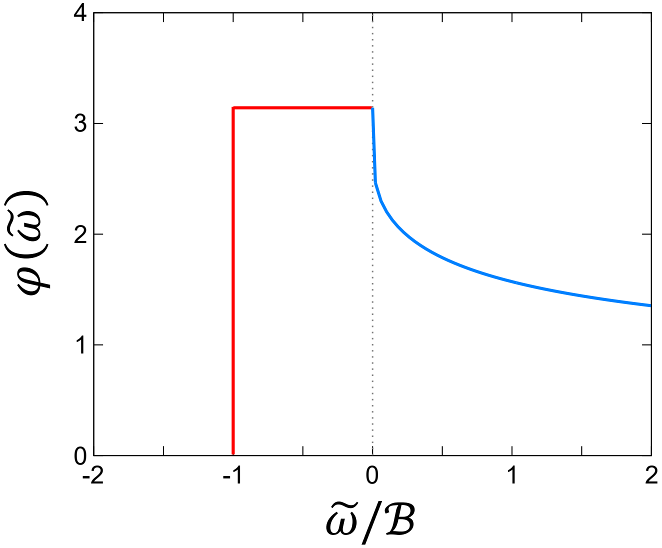
$$\mathcal{G}(K,\omega) = \frac{\mathcal{G}_0(K,\omega)}{1 - V\mathcal{G}_0(K,\omega)} \qquad \mathcal{G}_0(K,\omega) \simeq -\frac{m}{2\sqrt{3}\pi}\ln\left(\frac{-\tilde{\omega} - i\delta + \Lambda^2/m}{-\tilde{\omega} - i\delta}\right)$$

*Mott effect (medium suppression of bound state at small K) is neglected

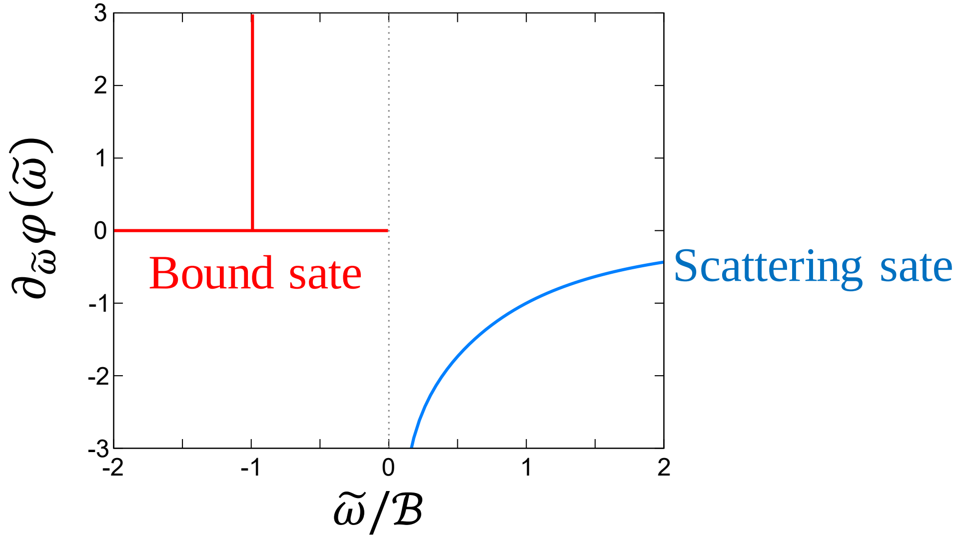
Derivative of the phase shift

$$\partial_{\tilde{\omega}}\varphi(\tilde{\omega}) = \pi\delta(\tilde{\omega} + \mathcal{B}) - \Theta(\tilde{\omega})\frac{\pi}{\tilde{\omega}[\ln^2(\tilde{\omega}/\mathcal{B}) + \pi^2]}$$

Phase shift



Derivative of phase shift



Crossover equation of state and baryonic distribution functions

$\Omega = \Omega_{\mathrm{HF}} + \delta\Omega_3$ Ω_{HF}: Hartree-Fock contribution

Tripling fluctuations:

$$\delta\Omega_3 = -T\sum_K \ln\left[1 + e^{-(-\mathcal{B}+E_{\mathrm{B}}^{\mathrm{kin}}-\tilde{\mu}_{\mathrm{B}})/T}\right]$$

$$+T\sum_K \int_0^\infty \frac{d\omega}{\omega} \frac{\ln\left[1+e^{-(\omega+E_{\mathrm{B}}^{\mathrm{kin}}-\tilde{\mu}_{\mathrm{B}})/T}\right]}{\ln^2(\omega/\mathcal{B})+\pi^2}$$

Baryonic distribution: $f_{\mathrm{B}}(K) = f(-\mathcal{B} + E_{\mathrm{B}}^{\mathrm{kin}} - \tilde{\mu}_{\mathrm{B}})$ **:Bound state**

$$-\frac{\partial\delta\Omega_3}{\partial\mu} = 3\sum_K f_{\mathrm{B}}(K)$$

$$-\int_0^\infty \frac{d\omega}{\omega} \frac{f(\omega+E_{\mathrm{B}}^{\mathrm{kin}}-\tilde{\mu}_{\mathrm{B}})}{\ln^2(\omega/\mathcal{B})+\pi^2}$$ **:Scattering state**

Baryon kinetic energy:

$$E_{\mathrm{B}}^{\mathrm{kin}}(K) = K^2/2M_{\mathrm{B}} \equiv K^2/(6m)$$

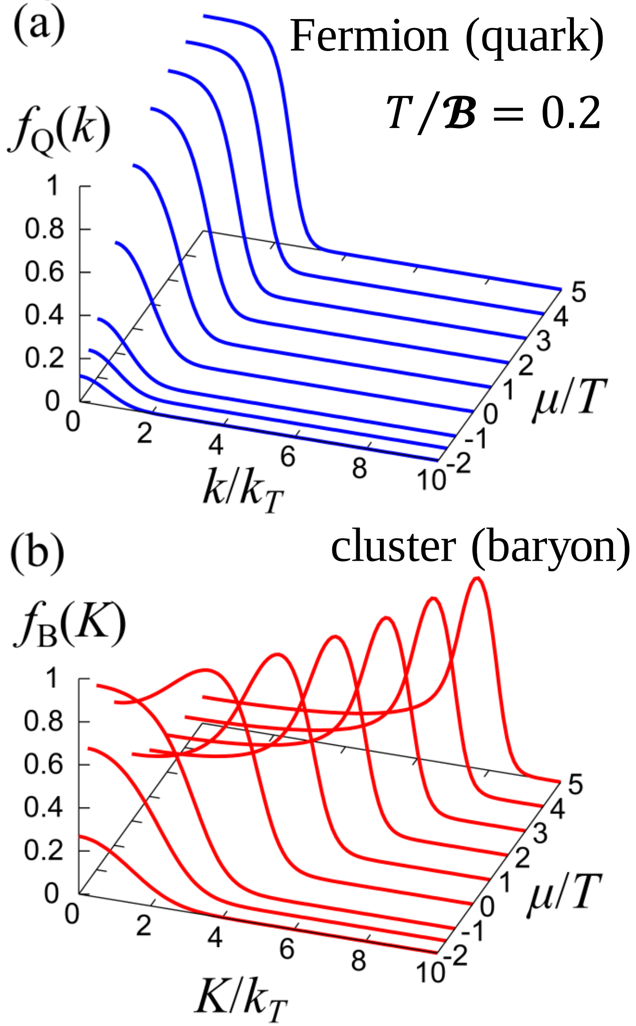
Baryon chemical potential:

$$\tilde{\mu}_{\mathrm{B}} = 3\tilde{\mu} \equiv 3(\mu - \Sigma_{\mathrm{HF}})$$

Momentum distributions

Model: 1D non-relativistic three-color fermions with color-singlet three-body interaction

Tripling fluctuation theory (present work)



$k_T = \sqrt{2mT}$: Thermal momentum scale

Momentum distributions

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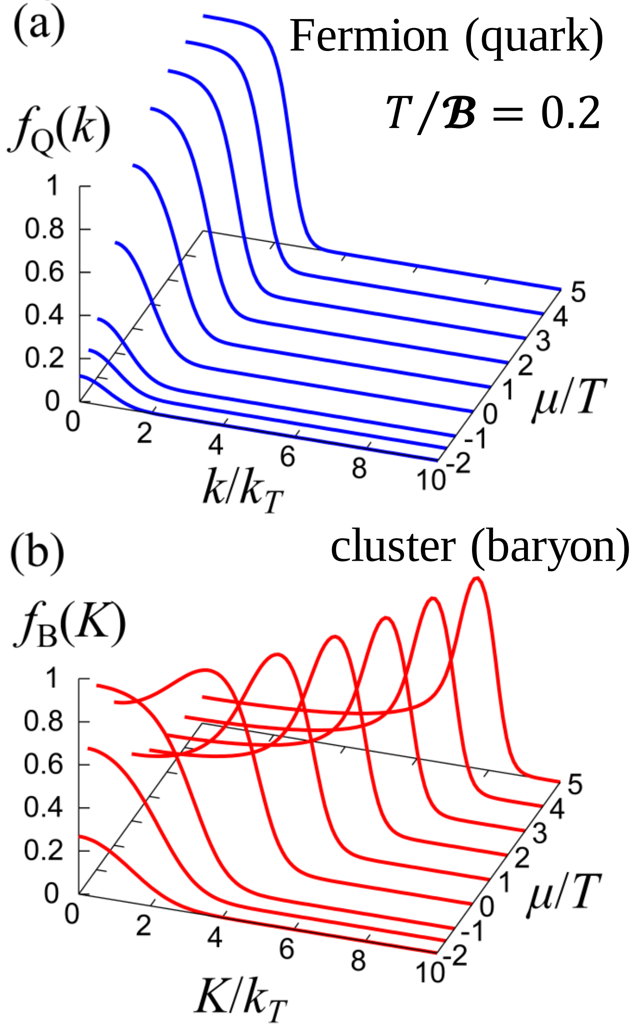
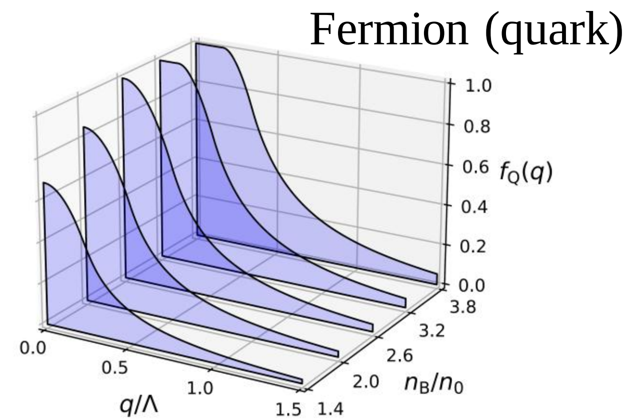
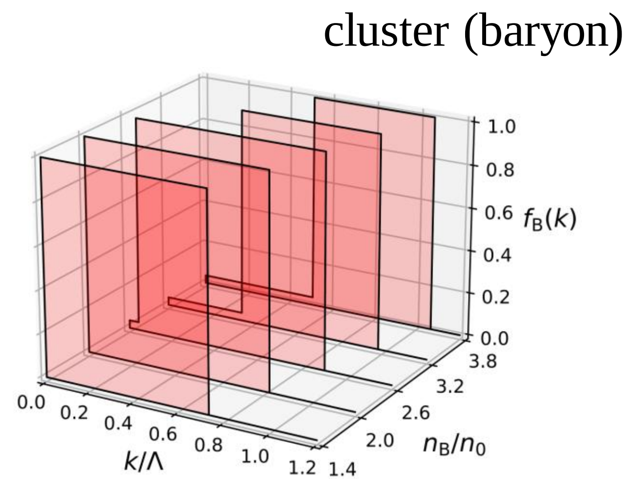


$k_T = \sqrt{2mT}$: Thermal momentum scale

Explicit Duality model

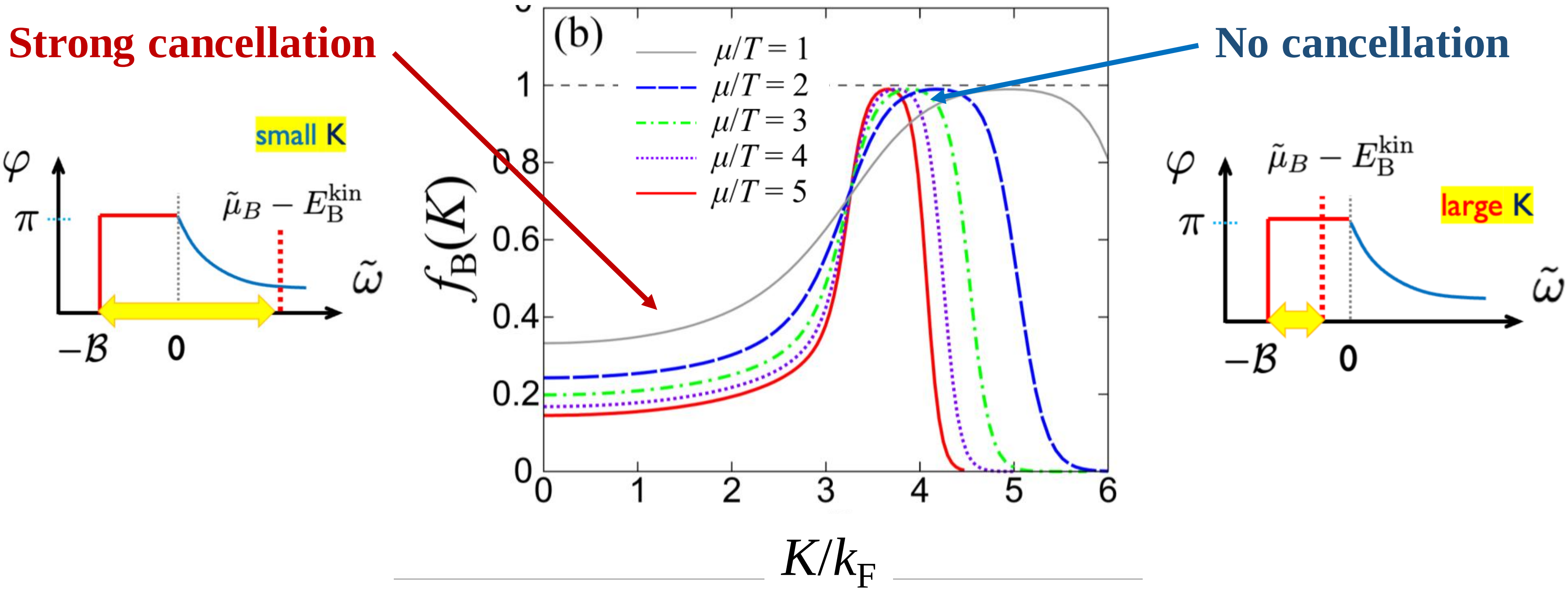
PRL **132**, 112701 (2024).





Baryonic momentum shell

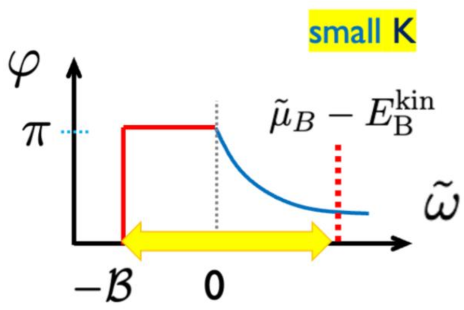
Baryon-like trimer momentum distribution

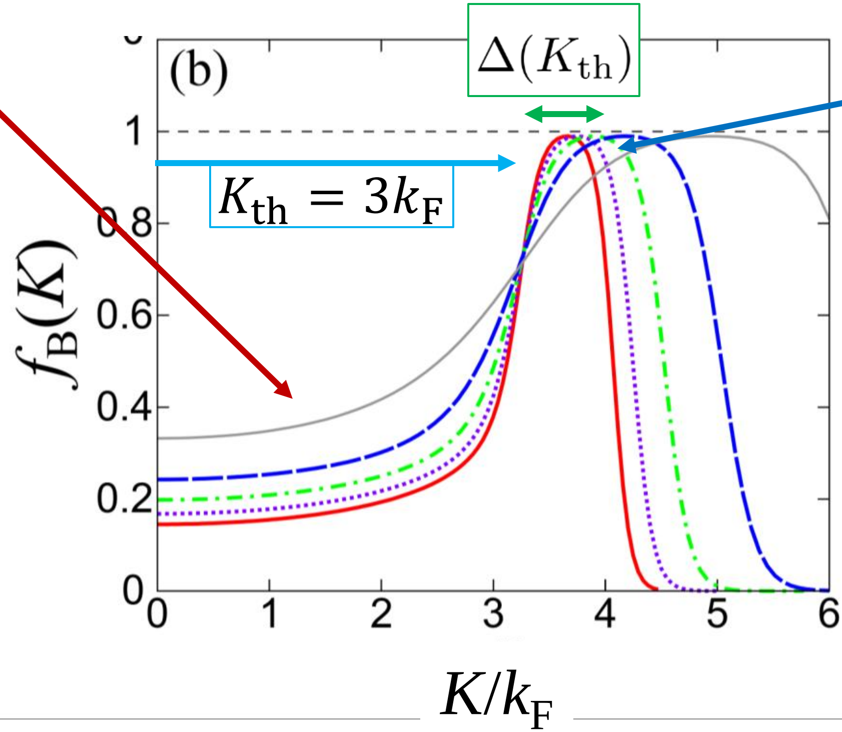


Baryonic momentum shell

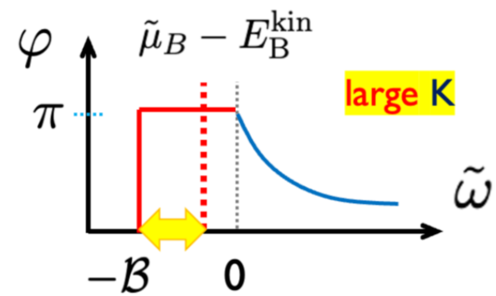
Baryon-like trimer momentum distribution

Strong cancellation





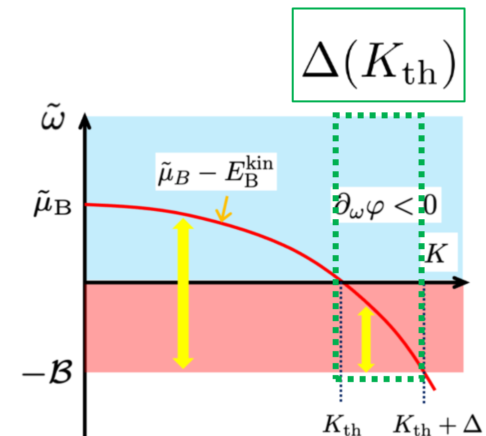
No cancellation



Momentum-shell width: $\Delta(K_{\rm th}) \equiv \sqrt{K_{\rm th}^2 + 2M_{\rm B}\mathcal{B}} - K_{\rm th}$

Analytical expression: $(T \to 0)$

$$f_{\rm B}^{T=0}(K) = \Theta(K_{\rm th} + \Delta - K)\Theta(K - K_{\rm th}) + \Theta(K_{\rm th} - K)\left(\frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left[\frac{\ln\frac{K_{\rm th}^2 - K^2}{2M_{\rm B}\mathcal{B}}}{\pi}\right]\right)$$



Peaked speed of sound

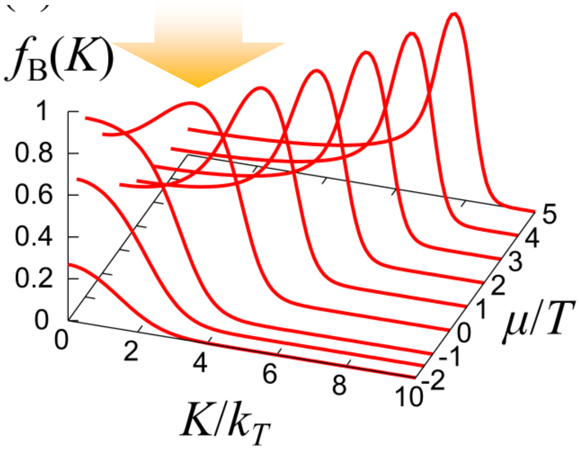
Squared speed of sound:

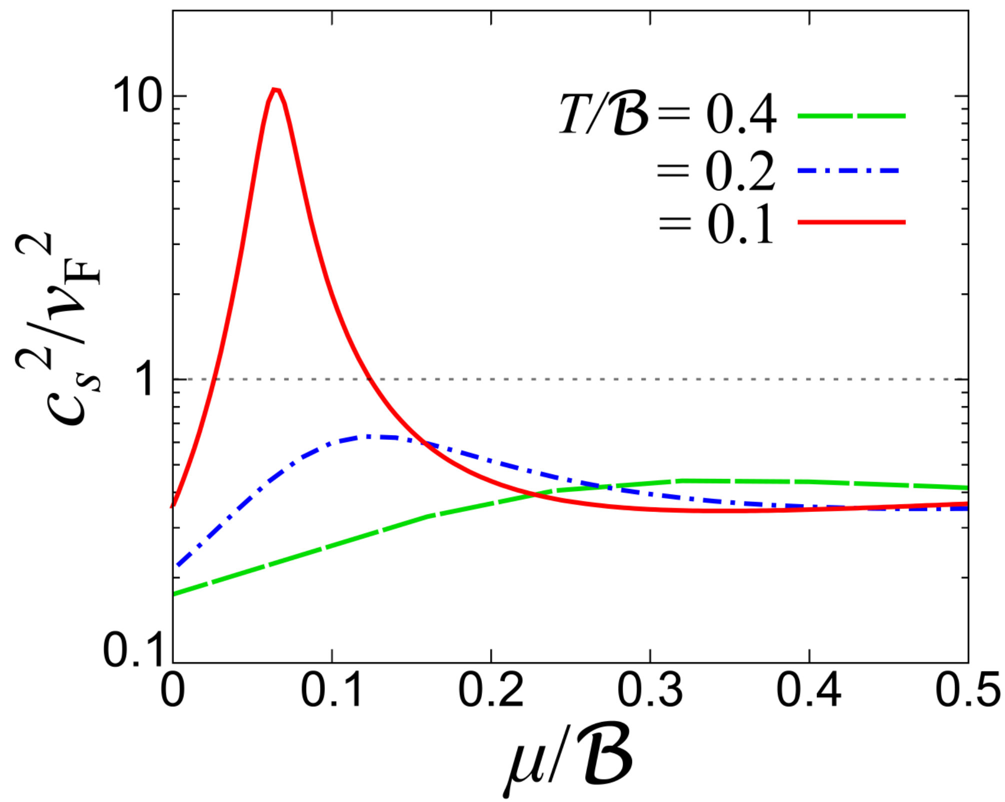
$$c_s^2 = \frac{n}{m}\left(\frac{\partial n}{\partial \mu}\right)^{-1}$$

Density susceptibility:

$$\frac{\partial n}{\partial \mu} = \frac{\partial n_{\mathrm{Q}}}{\partial \mu} + \frac{\partial n_{\mathrm{B}}}{\partial \mu}$$

$$\frac{\partial n_{\mathrm{B}}}{\partial \mu} = 3\sum_K \frac{\partial f_{\mathrm{B}}(K)}{\partial \mu} < 0$$



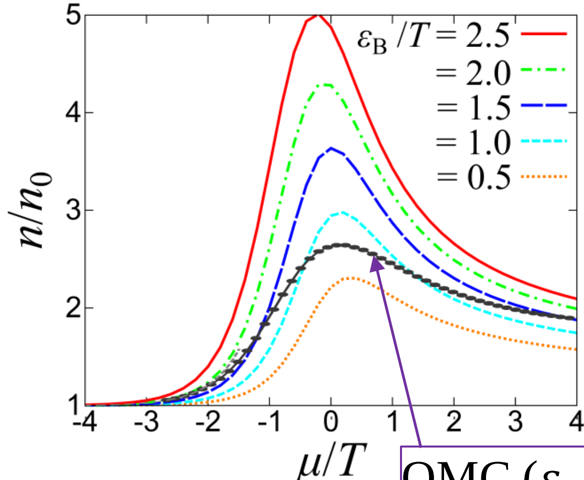


Peaked speed of sound is induced by suppressed baryon distributions at low momenta

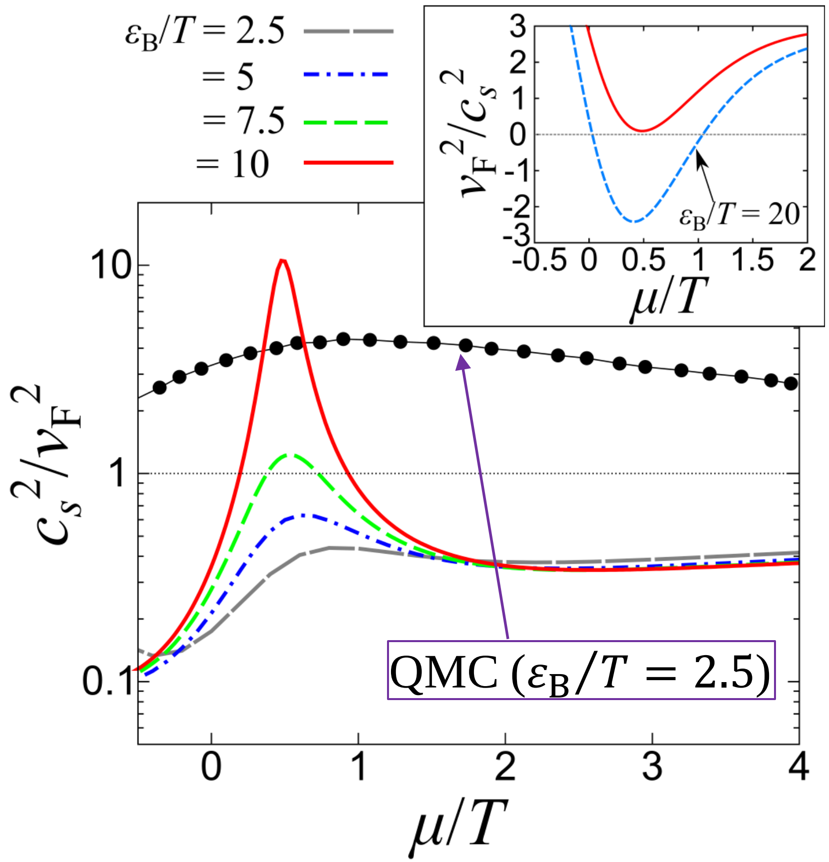
Comparison with QMC

Qualitatively OK, but better approximation is needed for quantitative calculation

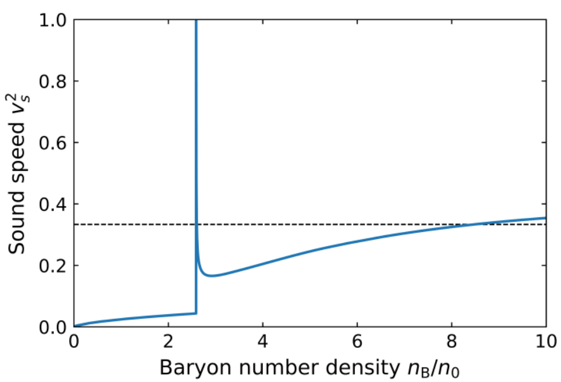
Density equation of state



Speed of sound



Speed of sound in the duality model



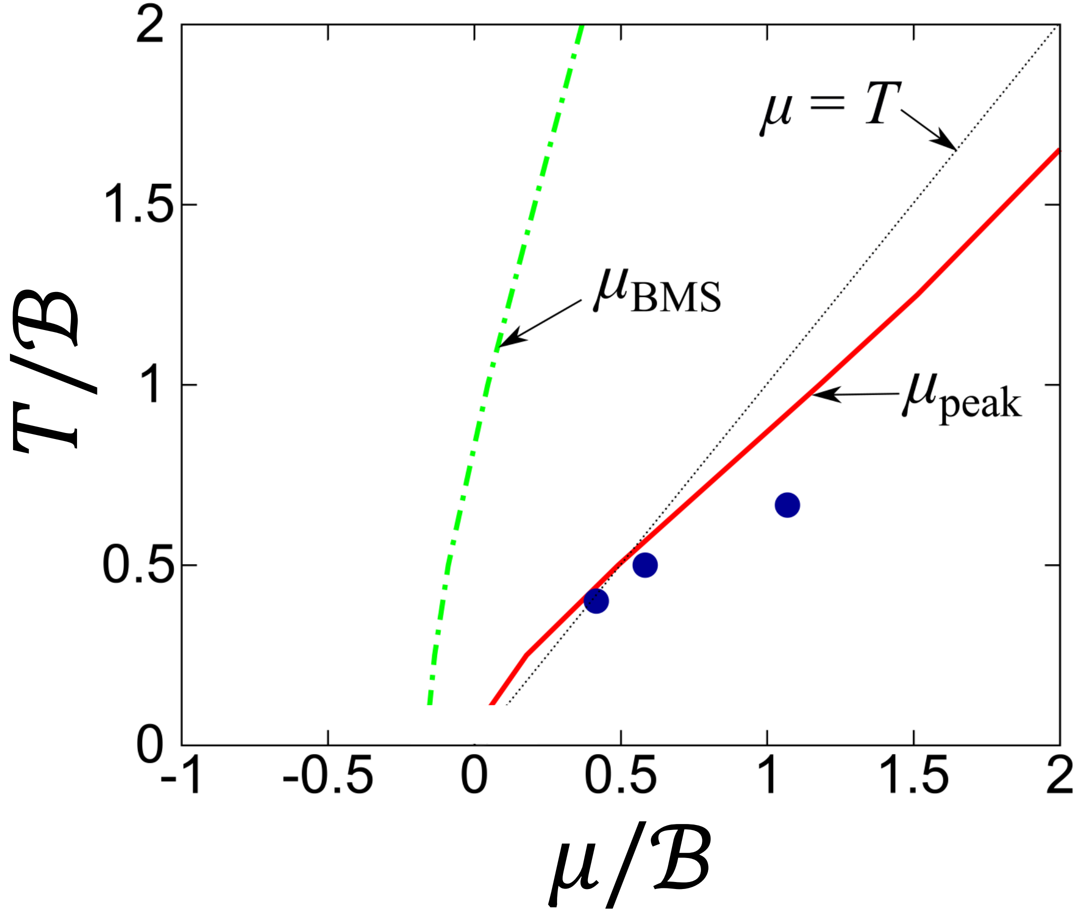
Y. Fujimoto, PRL **132**, 112701 (2024).

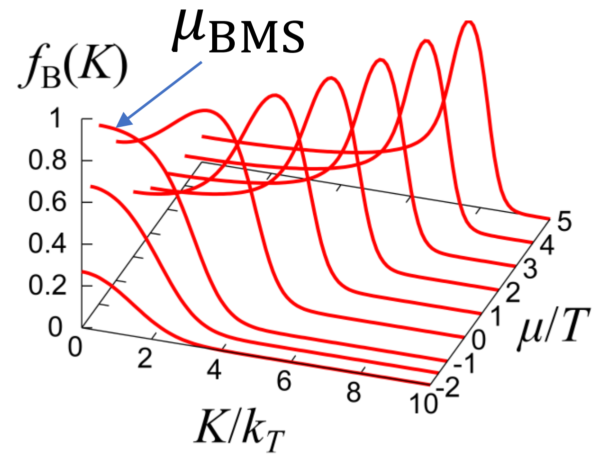
QMC: J. McKenny, *et al*., Phys. Rev. A **102**, 023313 (2020).

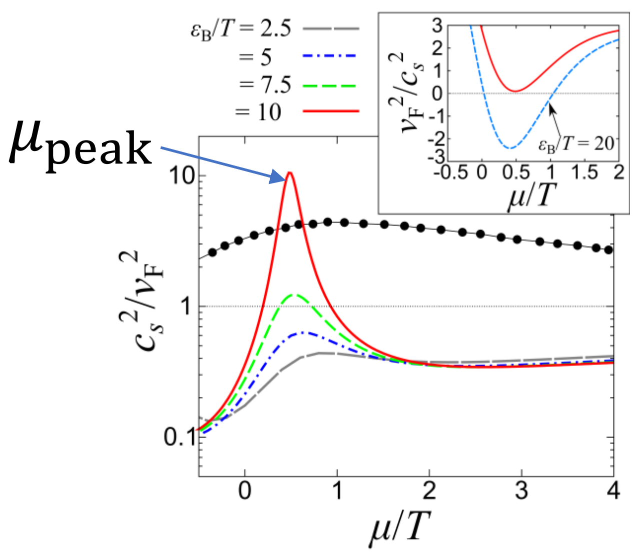
Finite-temperature phase diagram

μ_{BMS} : baryonic momentum shell starts to appear

μ_{peak} : sound velocity is peaked







● QMC: J. McKenny, *et al.*, Phys. Rev. A **102**, 023313 (2020).

Three-body force in ultracold atoms?

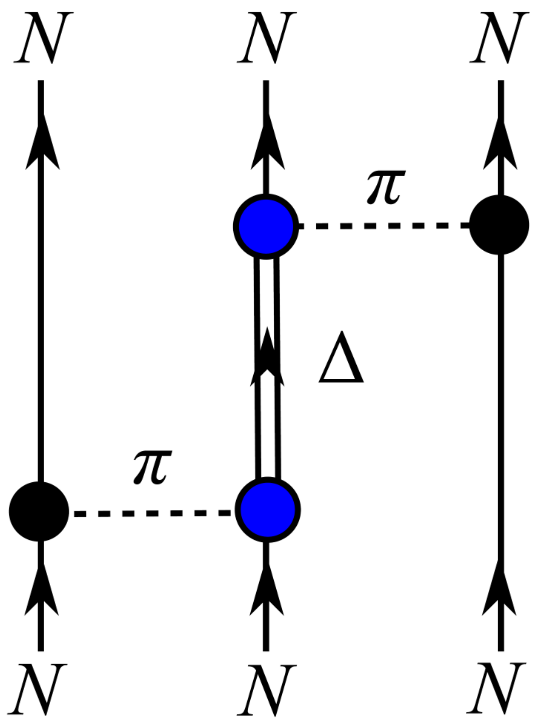
HT, E. Nakano, and K. Iida, arXiv:2505.19117

Nucleon ⇔

Pion ⇔

Δ resonance ⇔

Fujita-Miyazawa three-body force

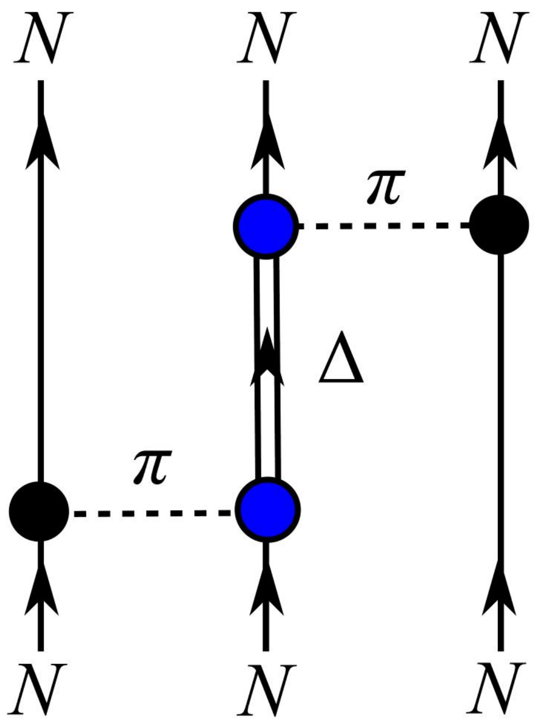


Three-body force in ultracold atoms?

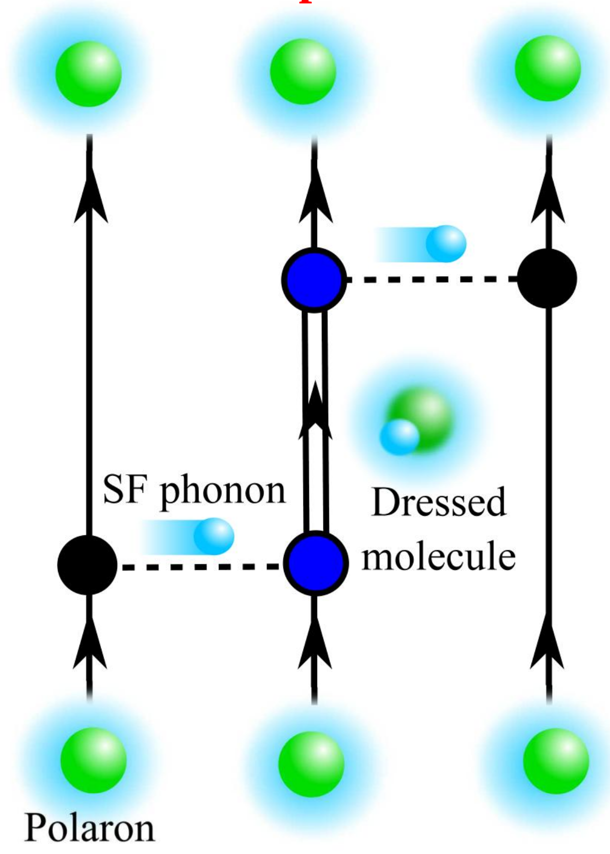
HT, E. Nakano, and K. Iida, arXiv:2505.19117

Nucleon ⇔ polaron (particle immersed in BEC)
Pion ⇔ superfluid phonon
Δ resonance ⇔ Feshbach molecule (closed-channel)

Fujita-Miyazawa three-body force



Tunable counterpart in ultracold atoms



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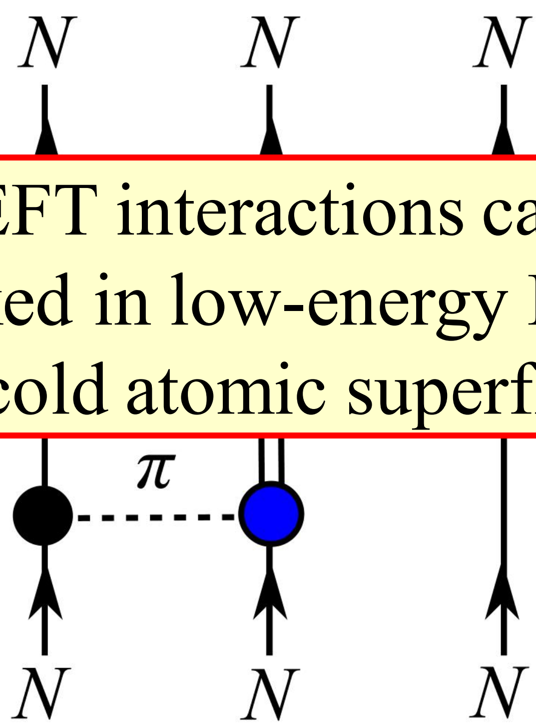
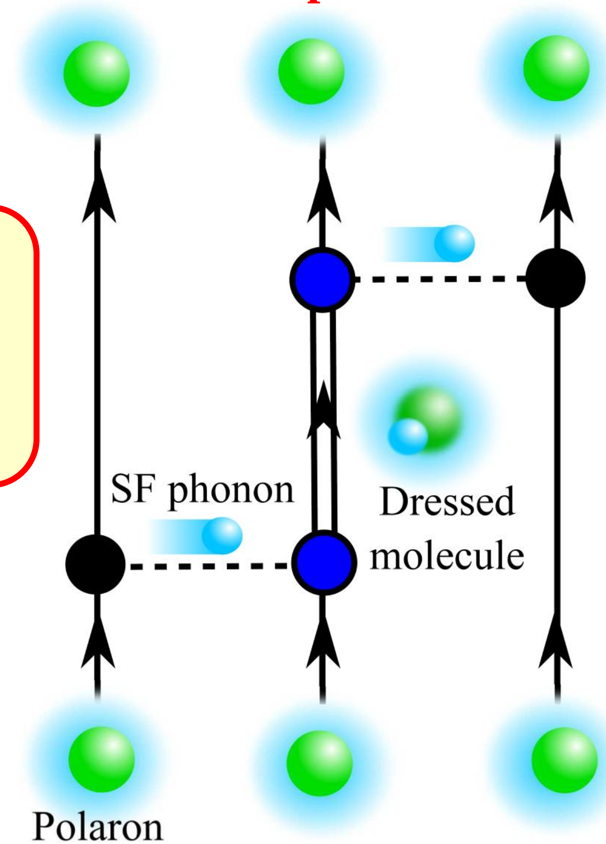
Fujita-Miyazawa three-body force



Tunable counterpart in ultracold atoms



Chiral EFT interactions can be mimicked in low-energy EFT of ultracold atomic superfluid!

Outline

- **Introduction**

  Can we study a microscopic mechanism of hadron-quark crossover in cold atom physics?

- **Formulation**

  Tripling fluctuation theory

- **Results**

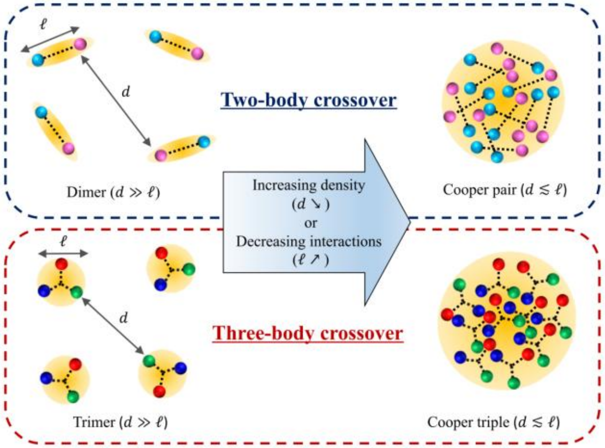
  Equation of state and momentum distributions

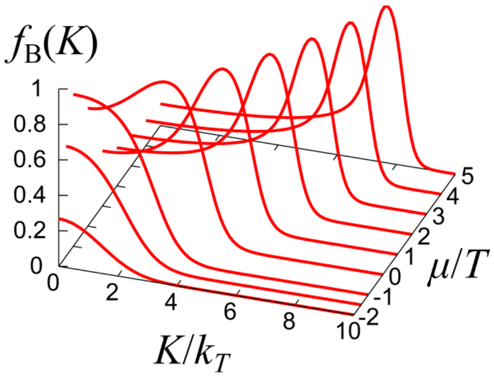
- **Summary**

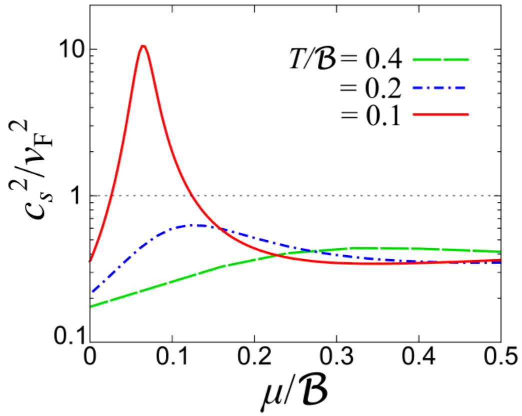
Summary of part 8

HT, K. Iida, T. Kojo, and H. Liang, PRL **135**, 042701 (2025).

- In analogy with the BEC-BCS crossover in two-component Fermi gases, we have discussed the three-body counterpart in three-color fermions, where bound trimer gases change into degenerate Fermi state with tripling fluctuations.
- It is found that tripling fluctuations can induce a peaked speed of sound as well as quarkyonic-like momentum distributions.





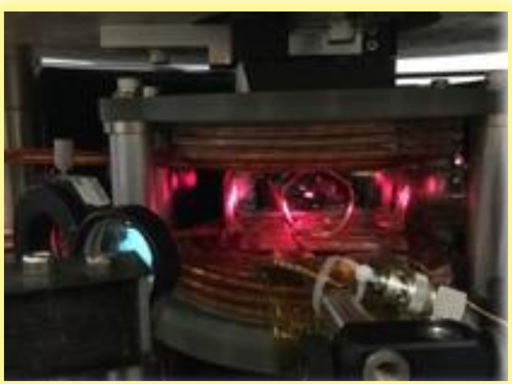


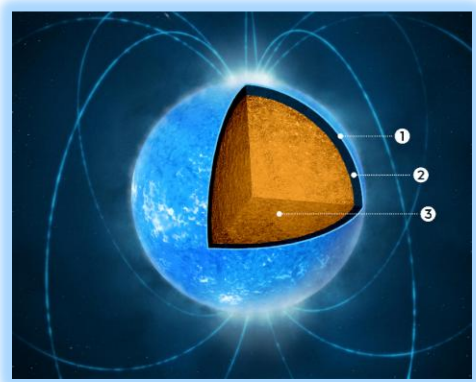
Future perspectives: Application to more realistic systems relevant to neutron-star matter, Bose-Fermi mixture, quantitative comparison with Monte Carlo simulation

Summary of the lecture

- We have discussed the interdisciplinary perspective on ultracold atoms and nuclear matter.
- Focus on pairing phenomena, BCS-BEC crossover, unitary Fermi gas, nucleon superfluid
- Success and failure of mean field theory, and how to go beyond within the diagrammatic approach
- Finally, we go beyond pairing and discussed tripling fluctuations in the hadron-quark crossover

Interdisciplinary scientific communications might lead to new discovery!





Thank you very much for your attention!

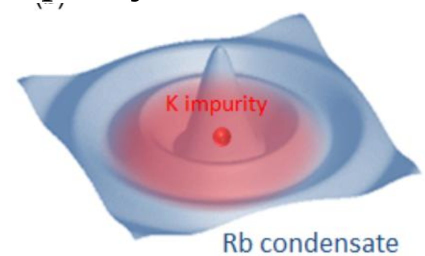
Appendix

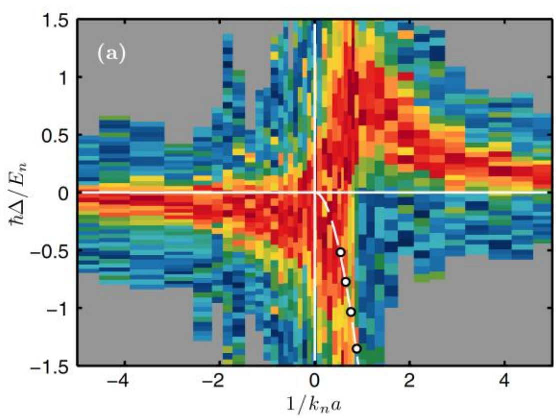
Yukawa interaction in cold atoms

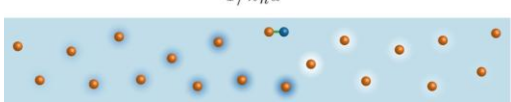
Two-polaron interaction in BEC is induced by exchange of superfluid phonons (analogous to pion exchange)

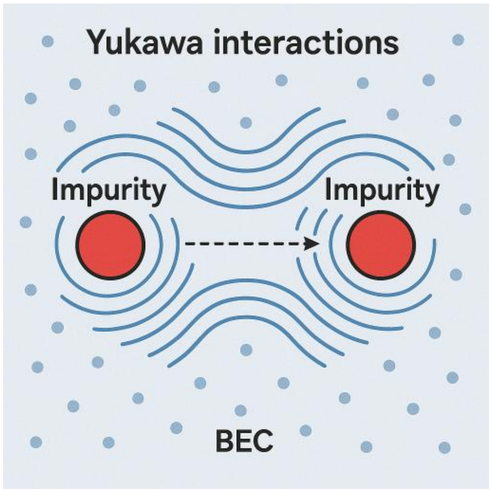
“Bose” polaron

Impurity immersed in BEC









by Chat GPT画伯

$$V_{2b}(r) = -\frac{\alpha}{r} e^{-r/\xi}$$

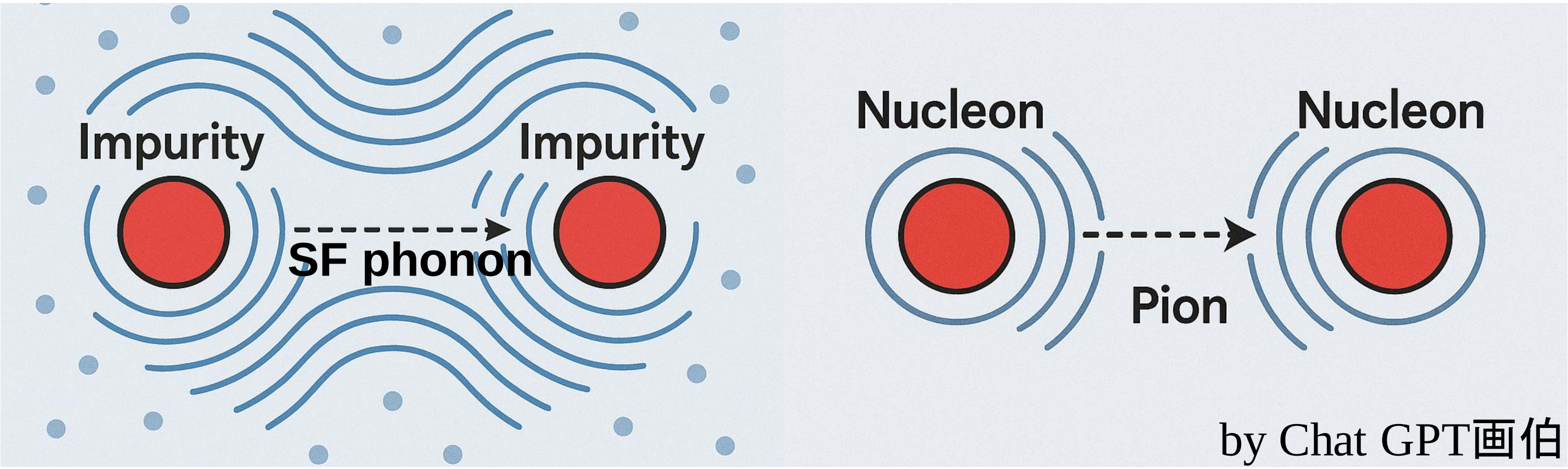
ξ: BEC healing length

Nils B. Jørgensen, et al., PRL **117**, 055302 (2016).

M. J. Mijslma, et al., PRA **61**, 053601 (2000).

Analogy between polaron and nucleon

Inter-polaron force **Nuclear force**



$$V_{2\mathrm{b}}(r) = -\frac{\alpha}{r} e^{-r/\xi}$$

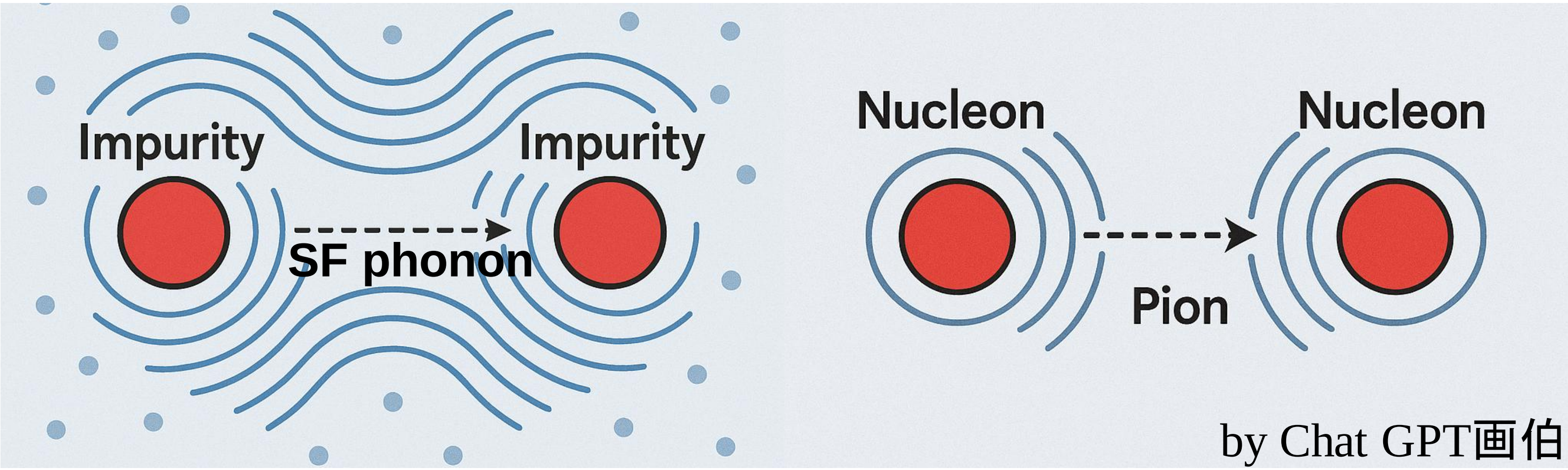
ξ: BEC healing length

$$V_{2\mathrm{b}}(r) = -\frac{\alpha'}{r} e^{-m_\pi r}$$

m_π^{-1}: inverse pion mass

Analogy between polaron and nucleon

Inter-polaron force **Nuclear force**



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$$V_{2b}(r) = -\frac{\alpha'}{r} e^{-m_\pi r}$$

ξ: BEC healing length

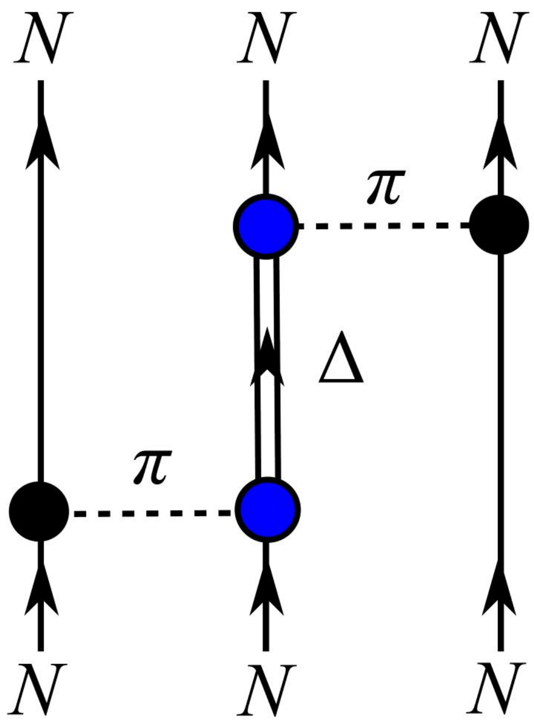
m_π^{-1}: inverse pion mass

Fujita-Miyazawa counterpart in three-polaron force?

Fujita-Miyazawa-type three-body force among polarons

HT, E. Nakano, and K. Iida, arXiv:2505.19117

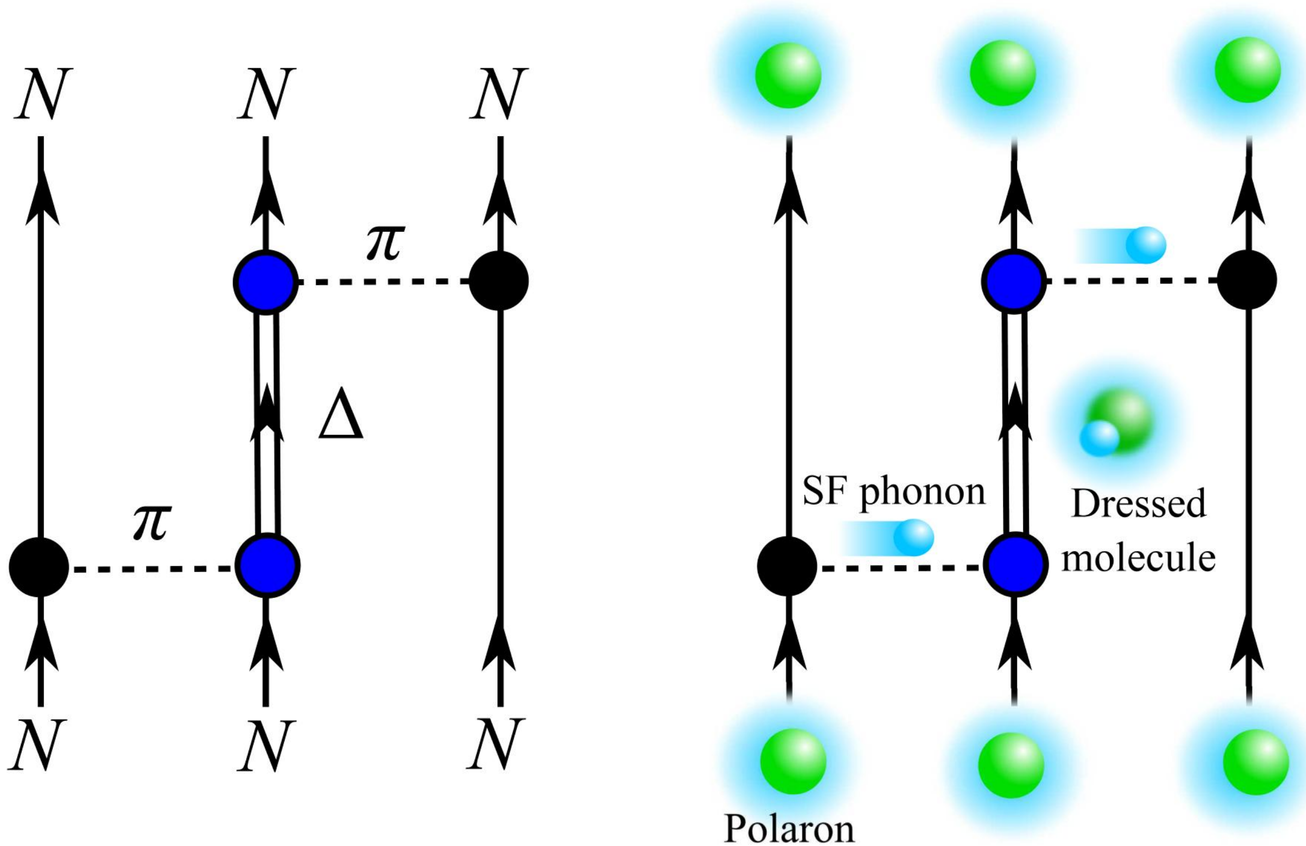
Nucleon
Pion
Δ resonance



Fujita-Miyazawa-type three-body force among polarons

HT, E. Nakano, and K. Iida, arXiv:2505.19117

Nucleon ⇔ polaron
Pion ⇔ superfluid phonon
Δ resonance ⇔ Feshbach molecule (closed-channel)



Two-channel model of cold atoms near the Feshbach resonance

$$\hat{H} = \sum_{\boldsymbol{k}} \left[\xi_{\boldsymbol{k},b} \hat{b}^\dagger_{\boldsymbol{k}} \hat{b}_{\boldsymbol{k}} + \xi_{\boldsymbol{k},c} \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} \right] + \sum_{\boldsymbol{P}} \xi_{\boldsymbol{P},A} \hat{A}^\dagger_{\boldsymbol{P}} \hat{A}_{\boldsymbol{P}}$$

$$+ \frac{U_{bb}}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^\dagger_{\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}^\dagger_{-\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}_{-\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}} \hat{b}_{\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}}$$

$$+ U_{bc} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^\dagger_{\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}^\dagger_{-\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} \hat{c}_{-\boldsymbol{k}'+\frac{M_c}{M_A}\boldsymbol{P}} \hat{b}_{\boldsymbol{k}'+\frac{M_b}{M_A}\boldsymbol{P}}$$

$$+ g \sum_{\boldsymbol{P},\boldsymbol{k}} \left[\hat{A}^\dagger_{\boldsymbol{P}} \hat{b}_{-\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}_{\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} + \text{h.c.} \right]$$

Kinetic energies:

$$\begin{aligned} \xi_{\boldsymbol{k},b} &= k^2/(2M_b) - \mu_b \\ \xi_{\boldsymbol{k},c} &= k^2/(2M_c) - \mu_c \\ \xi_{\boldsymbol{P},A} &= P^2/(2M_A) - \mu_b - \mu_c + \nu \end{aligned}$$

$\mu_{b,c}$: chemical potential

$M_{b,c,A}$: mass

$\nu(B)$: closed-channel energy

Two-channel model of cold atoms near the Feshbach resonance

Medium boson **Impurity** **Closed-channel molecule**

$$
\begin{aligned}
\hat{H} = & \sum_{\boldsymbol{k}} \left[\xi_{\boldsymbol{k},b} \hat{b}^{\dagger}_{\boldsymbol{k}} \hat{b}_{\boldsymbol{k}} + \xi_{\boldsymbol{k},c} \hat{c}^{\dagger}_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} \right] + \sum_{\boldsymbol{P}} \xi_{\boldsymbol{P},A} \hat{A}^{\dagger}_{\boldsymbol{P}} \hat{A}_{\boldsymbol{P}} \\
& + \frac{U_{bb}}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^{\dagger}_{\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}^{\dagger}_{-\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}_{-\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}} \hat{b}_{\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}} \\
& + U_{bc} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^{\dagger}_{\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}^{\dagger}_{-\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} \hat{c}_{-\boldsymbol{k}'+\frac{M_c}{M_A}\boldsymbol{P}} \hat{b}_{\boldsymbol{k}'+\frac{M_b}{M_A}\boldsymbol{P}} \\
& + g \sum_{\boldsymbol{P},\boldsymbol{k}} \left[\hat{A}^{\dagger}_{\boldsymbol{P}} \hat{b}_{-\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}_{\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} + \text{h.c.} \right]
\end{aligned}
$$

Kinetic energies:

$$
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\xi_{\boldsymbol{k},b} &= k^2/(2M_b) - \mu_b \\
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\end{aligned}
$$

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Two-channel model of cold atoms near the Feshbach resonance

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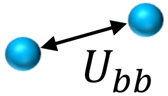
$$+ \frac{U_{bb}}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^\dagger_{\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}^\dagger_{-\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}_{-\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}} \hat{b}_{\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}}$$

Boson-boson interaction



$$+ U_{bc} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^\dagger_{\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}^\dagger_{-\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} \hat{c}_{-\boldsymbol{k}'+\frac{M_c}{M_A}\boldsymbol{P}} \hat{b}_{\boldsymbol{k}'+\frac{M_b}{M_A}\boldsymbol{P}}$$

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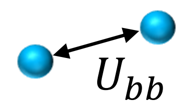
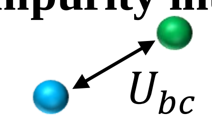
Boson-boson interaction



$$+ \frac{U_{bb}}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^{\dagger}_{\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}^{\dagger}_{-\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}_{-\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}} \hat{b}_{\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}}$$

Boson-impurity interaction



$$+ U_{bc} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^{\dagger}_{\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}^{\dagger}_{-\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} \hat{c}_{-\boldsymbol{k}'+\frac{M_c}{M_A}\boldsymbol{P}} \hat{b}_{\boldsymbol{k}'+\frac{M_b}{M_A}\boldsymbol{P}}$$

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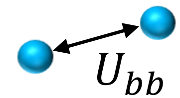
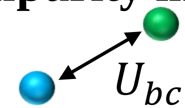
Boson-boson interaction

$$+ \frac{U_{bb}}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^{\dagger}_{\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}^{\dagger}_{-\boldsymbol{k}+\frac{\boldsymbol{P}}{2}} \hat{b}_{-\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}} \hat{b}_{\boldsymbol{k}'+\frac{\boldsymbol{P}}{2}}$$



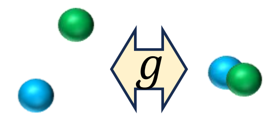
Boson-impurity interaction

$$+ U_{bc} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \hat{b}^{\dagger}_{\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}^{\dagger}_{-\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} \hat{c}_{-\boldsymbol{k}'+\frac{M_c}{M_A}\boldsymbol{P}} \hat{b}_{\boldsymbol{k}'+\frac{M_b}{M_A}\boldsymbol{P}}$$



Feshbach coupling

$$+ g \sum_{\boldsymbol{P},\boldsymbol{k}} \left[\hat{A}^{\dagger}_{\boldsymbol{P}} \hat{b}_{-\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}_{\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} + \text{h.c.} \right]$$



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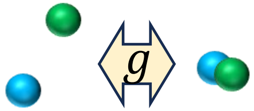
Polaron in Bose-Einstein condensate

$$\hat{b}_{\boldsymbol{k}} = \sqrt{n_0}\delta_{\boldsymbol{k},\mathbf{0}} + \hat{\pi}_{\boldsymbol{k}}(1 - \delta_{\boldsymbol{k},\mathbf{0}}),$$

n_0: BEC condensate density

Feshbach coupling

$$g \sum_{\boldsymbol{P},\boldsymbol{k}} \left[\hat{A}^{\dagger}_{\boldsymbol{P}} \hat{b}_{-\boldsymbol{k}+\frac{M_b}{M_A}\boldsymbol{P}} \hat{c}_{\boldsymbol{k}+\frac{M_c}{M_A}\boldsymbol{P}} + \text{h.c.} \right]$$



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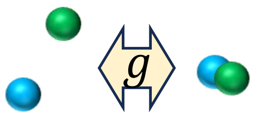
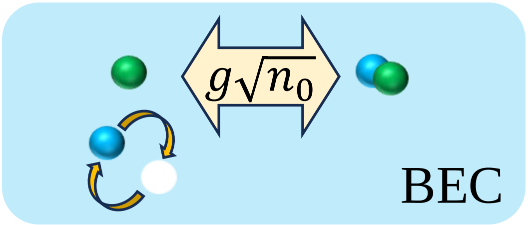


Coherent atom-molecule mixing

$$g\sqrt{n_0}\sum_{\boldsymbol{k}}\left(c^{\dagger}_{\boldsymbol{k}}A_{\boldsymbol{k}} + A^{\dagger}_{\boldsymbol{k}}c_{\boldsymbol{k}}\right)$$



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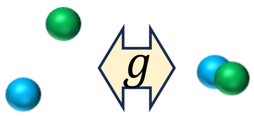
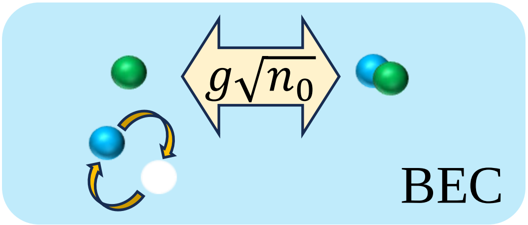


Coherent atom-molecule mixing

$$g\sqrt{n_0}\sum_{\boldsymbol{k}}\left(c^{\dagger}_{\boldsymbol{k}}A_{\boldsymbol{k}}+A^{\dagger}_{\boldsymbol{k}}c_{\boldsymbol{k}}\right)$$



Nucleon-like and Δ-like polarons as diagonalized eigenstates

$$\hat{H} = \hat{H}_N + \hat{H}_\Delta + \hat{H}_\pi + \hat{V}$$

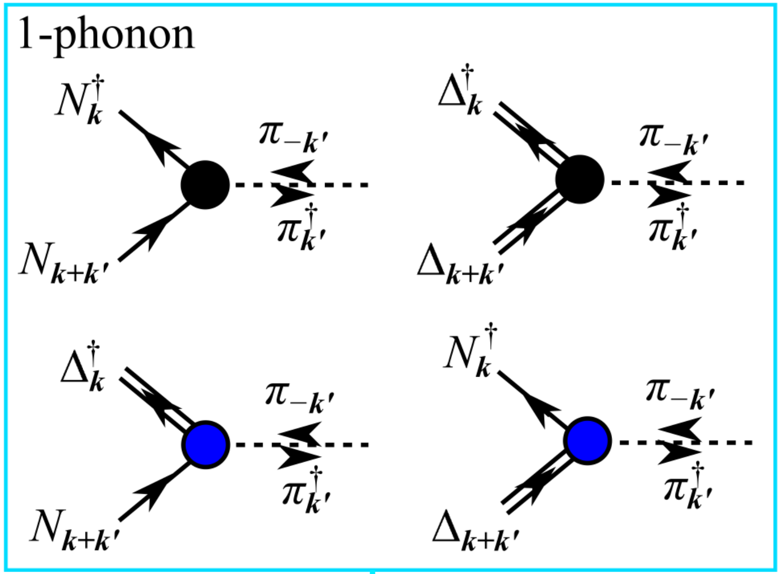
Ground state: $\hat{H}_N = \sum_{\boldsymbol{k}}\xi_{\boldsymbol{k},N}\hat{N}^{\dagger}_{\boldsymbol{k}}\hat{N}_{\boldsymbol{k}}$

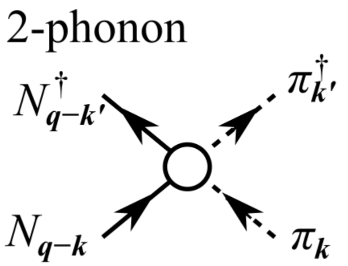
Excited state: $\hat{H}_\Delta = \sum_{\boldsymbol{k}}\xi_{\boldsymbol{k},\Delta}\hat{\Delta}^{\dagger}_{\boldsymbol{k}}\hat{\Delta}_{\boldsymbol{k}}$

Bogoliubov Hamiltonian for pion-like boson excitation

$$\hat{H}_\pi = \sum_{\boldsymbol{k}}(\xi_{\boldsymbol{k},b}+2U_{bb}n_0)\hat{\pi}^{\dagger}_{\boldsymbol{k}}\hat{\pi}_{\boldsymbol{k}} + \frac{U_{bb}n_0}{2}\sum_{\boldsymbol{k}}\left[\hat{\pi}^{\dagger}_{\boldsymbol{k}}\hat{\pi}^{\dagger}_{-\boldsymbol{k}}+\hat{\pi}_{-\boldsymbol{k}}\hat{\pi}_{\boldsymbol{k}}\right]$$

Absorption and emission of pion-like boson excitations





*Higher order at $n_0 \gg 1$

$$\hat{V} = \sum_{\boldsymbol{k},\boldsymbol{k}'} \Big[f^{NN\pi}_{\boldsymbol{k},\boldsymbol{k}'} \hat{N}^\dagger_{\boldsymbol{k}} \hat{N}_{\boldsymbol{k}+\boldsymbol{k}'} \hat{\pi}^\dagger_{\boldsymbol{k}'} + f^{\Delta\Delta\pi}_{\boldsymbol{k},\boldsymbol{k}'} \hat{\Delta}^\dagger_{\boldsymbol{k}} \hat{\Delta}_{\boldsymbol{k}+\boldsymbol{k}'} \hat{\pi}^\dagger_{\boldsymbol{k}'} + f^{\Delta N\pi}_{\boldsymbol{k},\boldsymbol{k}'} \hat{\Delta}^\dagger_{\boldsymbol{k}} \hat{N}_{\boldsymbol{k}+\boldsymbol{k}'} \hat{\pi}^\dagger_{\boldsymbol{k}'} + f^{N\Delta\pi}_{\boldsymbol{k},\boldsymbol{k}'} \hat{N}^\dagger_{\boldsymbol{k}} \hat{\Delta}_{\boldsymbol{k}+\boldsymbol{k}'} \hat{\pi}^\dagger_{\boldsymbol{k}'} \Big] + \text{h.c.}$$

Hamiltonian effective field theory based on the open-system description

We do not have to resort to path integral formalism

Grand-canonical partition function

$$Z = \mathrm{Tr}\left[e^{-\beta(\hat{H}_N + \hat{H}_\Delta + \hat{H}_\pi + \hat{V})}\right]$$

Hamiltonian effective field theory based on the open-system description

We do not have to resort to path integral formalism

Grand-canonical partition function

$$Z = \mathrm{Tr}\left[e^{-\beta(\hat{H}_N + \hat{H}_\Delta + \hat{H}_\pi + \hat{V})}\right]$$

"Trace out"

"Effective nucleon system"

$$Z = \mathrm{Tr}_N\left[e^{-\beta(\hat{H}_N + \hat{V}_{\mathrm{eff}})}\right]$$

$\hat{V}_{\mathrm{eff}}$: effective interaction

$\mathrm{Tr}_N[\ldots]$: partial trace of N state

Hamiltonian effective field theory based on the open-system description

We do not have to resort to path integral formalism

Grand-canonical partition function

$$Z = \mathrm{Tr}\left[e^{-\beta(\hat{H}_N+\hat{H}_\Delta+\hat{H}_\pi+\hat{V})}\right]$$

"Trace out"

"Effective nucleon system"

$$Z = \mathrm{Tr}_N\left[e^{-\beta(\hat{H}_N+\hat{V}_{\mathrm{eff}})}\right]$$

$\hat{V}_{\mathrm{eff}}$: effective interaction

$\mathrm{Tr}_N[\ldots]$: partial trace of N state

Equation for effective interaction

$$e^{-\beta\hat{V}_{\mathrm{eff}}} = \mathrm{Tr}_{\Delta\pi}\left[e^{-\beta(\hat{H}_\Delta+\hat{H}_\pi)}\hat{S}(\beta)\right]$$

S-matrix operator

$$\hat{S}(\beta) = T_\tau \exp\left[-\int_0^\beta d\tau\,\hat{V}(\tau)\right]$$

Interaction representation in the imaginary time formalism

$$\hat{V}(\tau) = e^{\tau(\hat{H}_N+\hat{H}_\Delta+\hat{H}_\pi)}\hat{V}e^{-\tau(\hat{H}_N+\hat{H}_\Delta+\hat{H}_\pi)}$$

Hamiltonian effective field theory based on the open-system description

We do not have to resort to path integral formalism

Grand-canonical partition function

$$Z = \mathrm{Tr}\left[e^{-\beta(\hat{H}_N + \hat{H}_\Delta + \hat{H}_\pi + \hat{V})}\right]$$

"Trace out"

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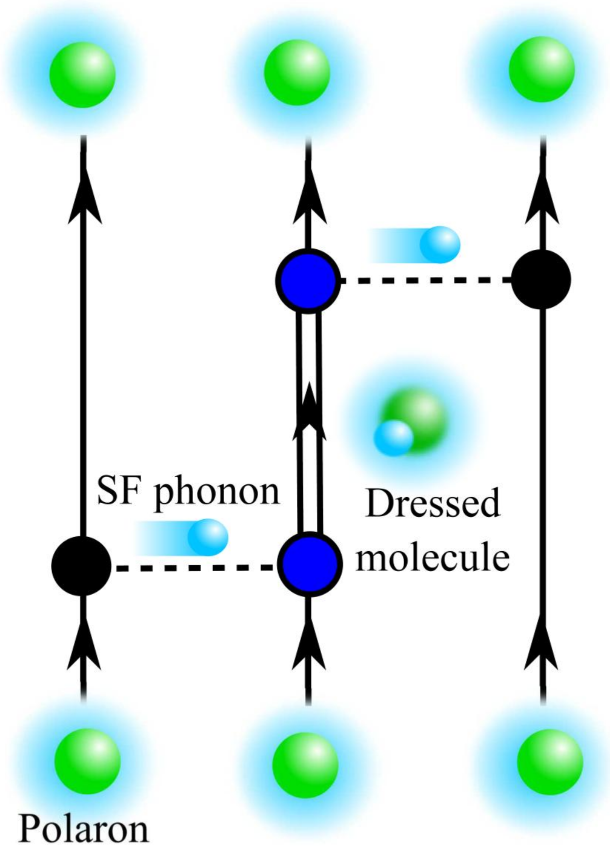
$$\hat{V}(\tau) = e^{\tau(\hat{H}_N + \hat{H}_\Delta + \hat{H}_\pi)}\hat{V}e^{-\tau(\hat{H}_N + \hat{H}_\Delta + \hat{H}_\pi)}$$

Perturbative expression of the effective interaction

$$\hat{V}_{\mathrm{eff}} = \sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}}{\ell!\beta}\int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_\ell \langle T_\tau[\hat{V}(\tau_1)\cdots\hat{V}(\tau_\ell)]\rangle$$

$$\langle\cdots\rangle = \mathrm{Tr}_{\pi\Delta}[e^{-\beta(\hat{H}_\Delta + \hat{H}_\pi)}\ldots]/\mathrm{Tr}_{\pi\Delta}[e^{-\beta(\hat{H}_\Delta + \hat{H}_\pi)}]$$

Fujita-Miyazawa three-body force



$$\hat{V}_{\rm FM} = \frac{1}{6}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{q}_1,\boldsymbol{q}_2} U_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}(\boldsymbol{q}_1,\boldsymbol{q}_2) \times \hat{N}^\dagger_{\boldsymbol{k}_1}\hat{N}^\dagger_{\boldsymbol{k}_2}\hat{N}^\dagger_{\boldsymbol{k}_3}\hat{N}_{\boldsymbol{k}_3-\boldsymbol{q}_1}\hat{N}_{\boldsymbol{k}_2+\boldsymbol{q}_1-\boldsymbol{q}_2}\hat{N}_{\boldsymbol{k}_1+\boldsymbol{q}_2}$$

2π-exchange-like form of coupling strength

$$U_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}(\boldsymbol{q}_1,\boldsymbol{q}_2) = -6\mathcal{G}^{\Delta}_{\boldsymbol{k}_2+\boldsymbol{q}_1}\mathcal{G}^{N\pi\Delta}_{\boldsymbol{k}_1,\boldsymbol{q}_2,\boldsymbol{k}_2+\boldsymbol{q}_1}\mathcal{G}^{\Delta\pi N}_{\boldsymbol{k}_2,\boldsymbol{q}_1,\boldsymbol{k}_3}$$

Δ prop. π-like SF phonon prop. with form factors

At $g \ll U_{bc}\sqrt{n_0}$

$$U_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}(\boldsymbol{q}_1,\boldsymbol{q}_2) \propto \frac{1}{(\boldsymbol{q}_1^2+\xi^{-2})(\boldsymbol{q}_2^2+\xi^{-2})}$$

ξ: BEC healing length

How to measure?

Interaction energy in the impurity equation of state

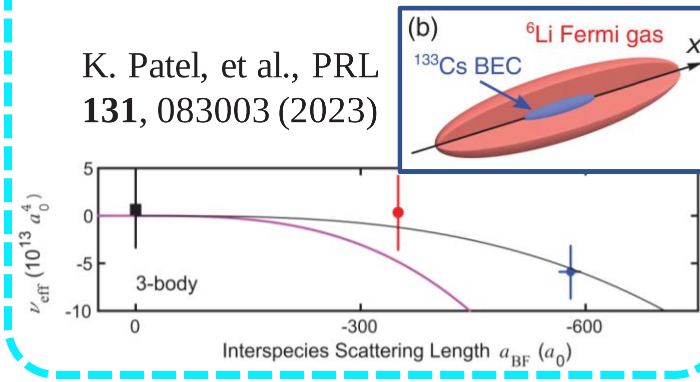
$$\delta E_3 \propto U_{\mathbf{0},\mathbf{0},\mathbf{0}}(\mathbf{0},\mathbf{0})n_N^3$$

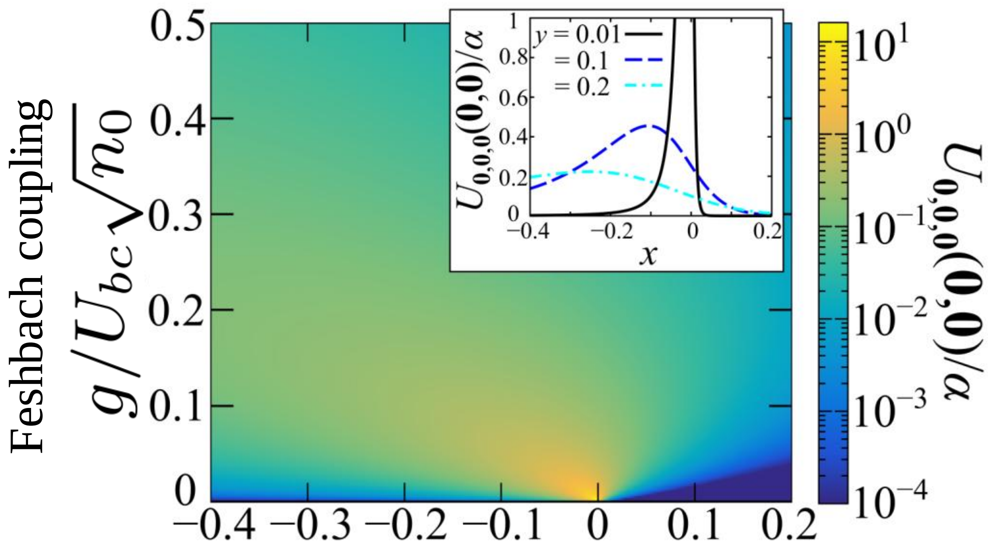
n_N: ground-state polaron density

$$U_{\mathbf{0},\mathbf{0},\mathbf{0}}(\mathbf{0},\mathbf{0}) = \frac{\alpha y^2 (1-x/2)^2}{(x^2+4y^2)^{3/2}} \left(\frac{1}{2} - \frac{y^2+x/2}{2\sqrt{x^2+4y^2}}\right)^2$$

Observation of fermion-mediated three-body force

K. Patel, et al., PRL **131**, 083003 (2023)





Tunable via $\nu(B)$!

Closed-channel energy level: $\{\nu - (U_{bc} + U_{bb})n_0\}/U_{bc}n_0$

Realization of tunable three-body interaction in cold atoms

A. Hammond, *et al.*, Phys. Rev. Lett. **128**, 083401 (2022)

EOS in Rabi-coupled 2-com. 1D BEC

$$\frac{E_{\rm MF}}{V} = -\frac{\hbar\Omega}{2}(\phi_\uparrow^*\phi_\downarrow + \phi_\downarrow^*\phi_\uparrow) + \frac{\hbar\delta}{2}(|\phi_\uparrow|^2 - |\phi_\downarrow|^2) + \sum_{\sigma\sigma'}\frac{g_{\sigma\sigma'}}{2}|\phi_\sigma|^2|\phi_{\sigma'}|^2.$$

Low-energy EFT

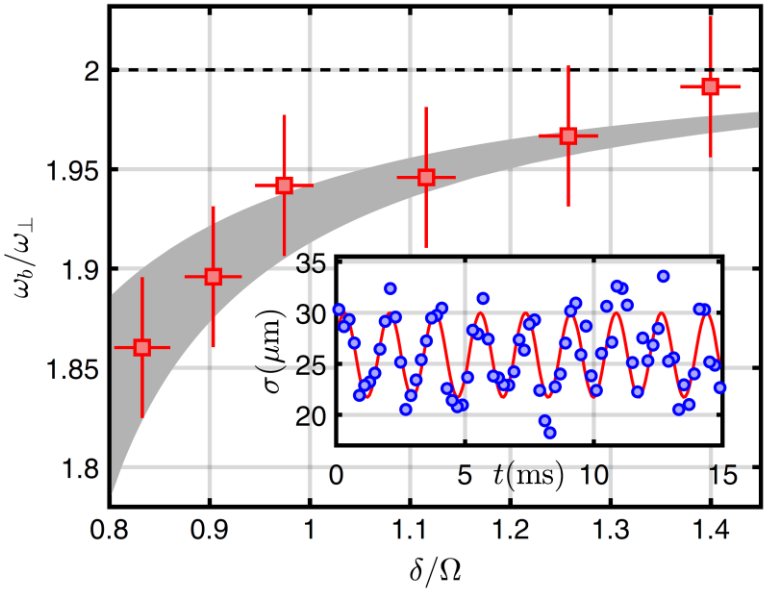
$$\frac{E_{\rm MF}}{N} \approx \epsilon_- + g_2\frac{n}{2} + g_3\frac{n^2}{3}$$

with $\quad g_2 = g - \dfrac{\bar{g}}{1+\delta^2/\Omega^2}$

and $\quad g_3 = -\dfrac{3\bar{g}^2}{\hbar\Omega}\dfrac{\delta^2/\Omega^2}{(1+\delta^2/\Omega^2)^{5/2}}$

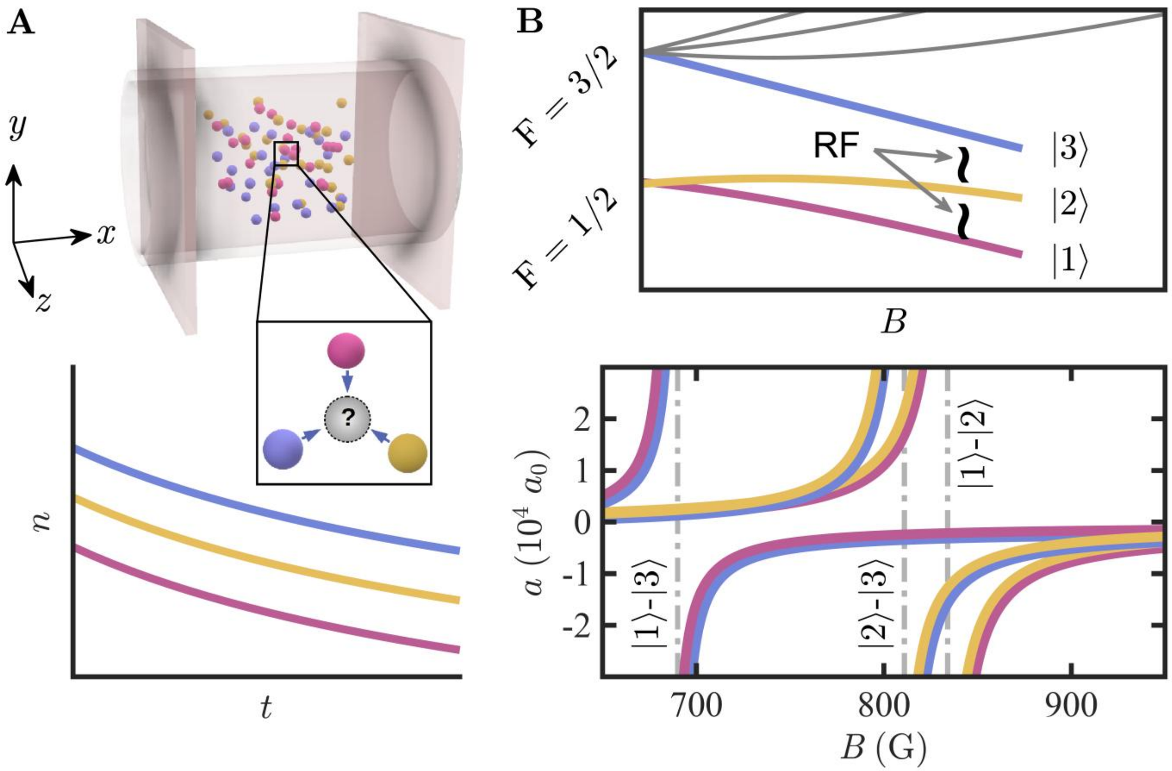
Breezing mode frequency

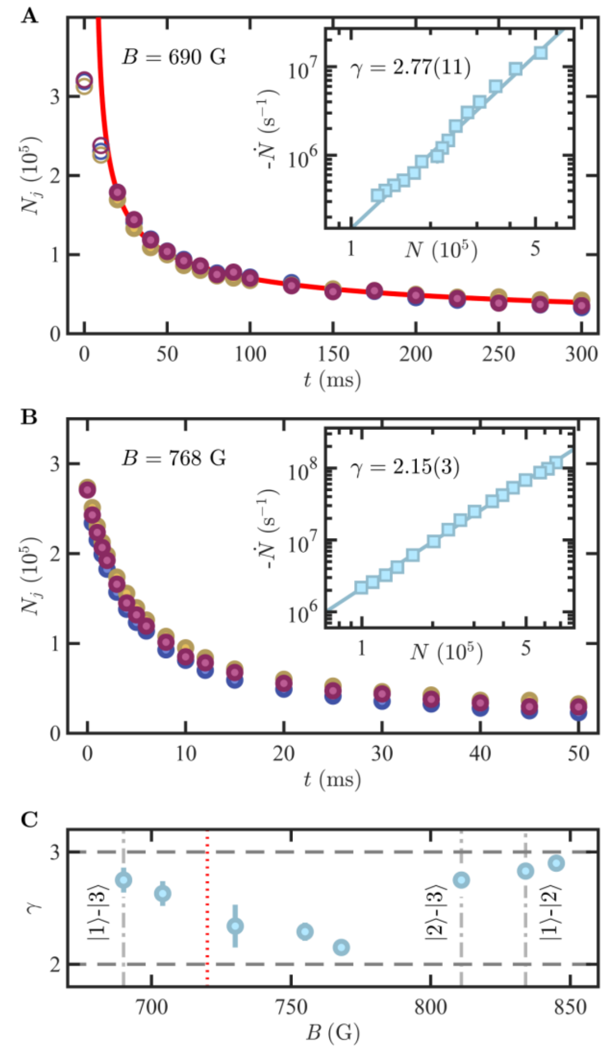
$$\omega_b = 2\omega_\perp\sqrt{1 + E_3/E_{\rm pot}}$$



Recent experiments of three-component Fermi gases

G. L. Schumacher, et al., arXiv:2301.92237





1D nonrelativistic three-color fermions with three-body interaction

- Hamiltonian density: $\hat{H} = \hat{H}_0 + \hat{V}_3$

One-body kinetic term

$$\hat{H}_0 = \sum_{a=\mathrm{r,g,b}} \psi_a^\dagger \left(-\frac{\partial_x^2}{2m} - \mu \right) \psi_a$$

μ: chemical potential

$a = \mathrm{r, g, b}$: pseudo-color (hyperfine states)

$\psi_a^\dagger, \psi_a$: fermionic field operator

Three-body interaction (classically scale invariant: $x \to \lambda^{-1} x$)

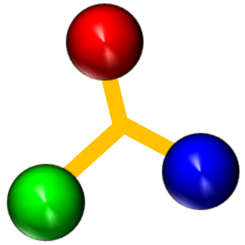
J. Drut, et al., PRL **120**, 243002 (2018).

$$\hat{V}_3 = V(\psi_\mathrm{r}^\dagger \psi_\mathrm{r})(\psi_\mathrm{g}^\dagger \psi_\mathrm{g})(\psi_\mathrm{b}^\dagger \psi_\mathrm{b})$$

$V < 0$: three-body attraction

Three-body binding energy (broken scale invariance)

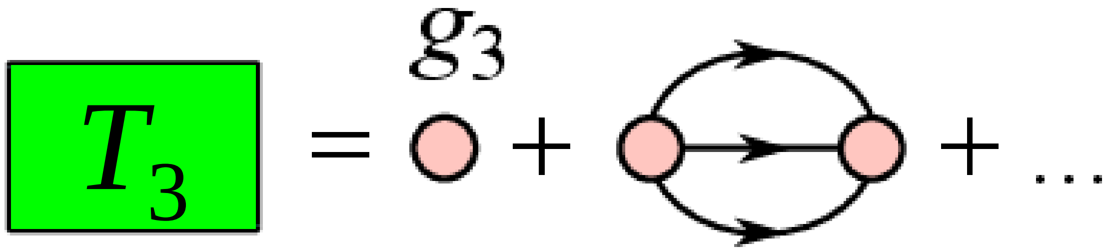
$$\mathcal{B} = \frac{\Lambda^2}{m} \exp\left(\frac{2\sqrt{3}\pi}{mV} \right)$$



Λ: UV cutoff scale

Three-body T-matrix for three-body interaction

Three-body coupling constant g_3 can be represented by the three-body binding energy ε_B



$$T_3(P,\Omega_+) = \left[\frac{1}{g_3} - \Xi_0(P,\Omega_+)\right]^{-1}$$

Ξ_0: Three-body propagator in vacuum

$$\Xi_0(P,\Omega_+) = \sum_{k,q} \frac{1}{\Omega_+ - \varepsilon_{\frac{P}{3}+k-\frac{q}{2}} - \varepsilon_{\frac{P}{3}+q} - \varepsilon_{\frac{P}{3}-k-\frac{q}{2}}} = -\frac{m}{2\sqrt{3}\pi}\ln\left(\frac{\Lambda^2 + P^2/6 - m\Omega_+}{P^2/6 - m\Omega_+}\right)$$

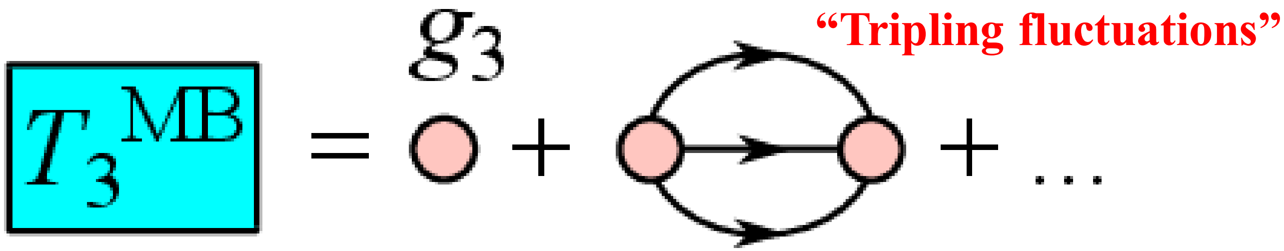
Λ: cutoff

<u>**Three-body binding energy**</u>

$$\frac{1}{g_3} - \Xi_0(0,\Omega = -\varepsilon_\mathrm{B}) = 0 \quad \Rightarrow \quad \varepsilon_\mathrm{B} = \frac{\Lambda^2}{m}\exp\left(\frac{2\sqrt{3}\pi}{m g_3}\right)$$

In-medium three-body T-matrix

HT, S. Tsutsui, T. M. Doi, and K. Iida, Phys. Rev. Research **4**, L012021 (2022).



$$T_3^{\mathrm{MB}}(\boldsymbol{P}, i\Omega_n) = \left[\frac{1}{g_3} - \Xi(\boldsymbol{P}, i\Omega_n)\right]^{-1}$$

Ξ: In-medium three-particle (three-hole) propagator

$\Omega_n = (2n+1)\pi T$: Fermion Matsubara frequency

In-medium three-body equation

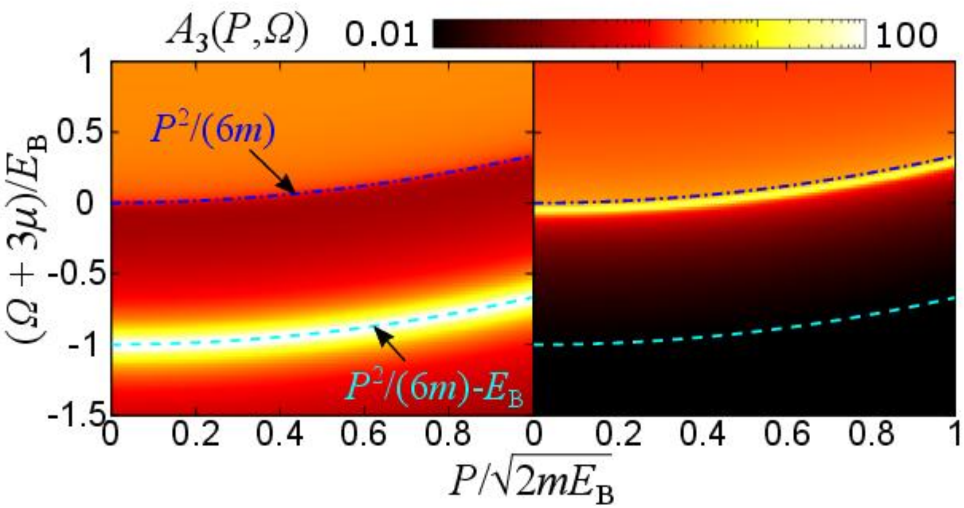
$$\frac{1}{g_3} - \Xi(\boldsymbol{P} = 0, \Omega = -E_{\mathrm{B}}^{\mathrm{M}}) = 0$$

Three-body spectral function

HT, S. Tsutsui, T. M. Doi, and K. Iida, Phys. Rev. Research **4**, L012021 (2022).

In-medium three-body spectra

$$A_3(P,\Omega) = -\mathrm{Im}T_3^{\mathrm{MB}}(P,\Omega_+)$$

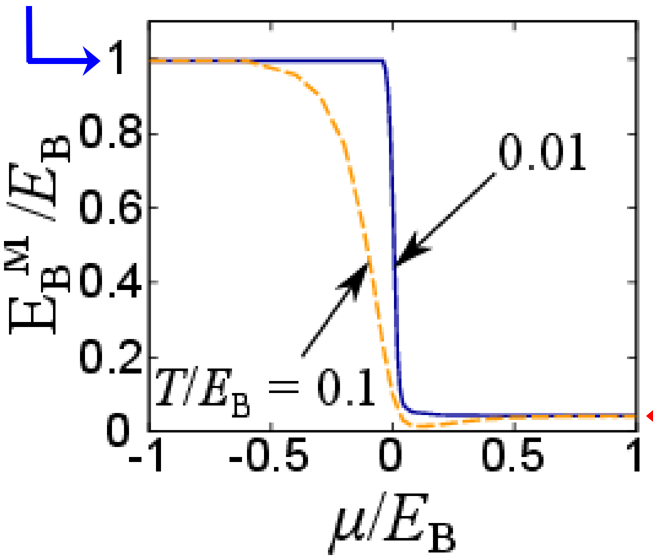


Low density ($\mu/E_B = -1$)

High-density ($\mu/E_B = 2$)

In-medium three-body binding energy

Three-body problem



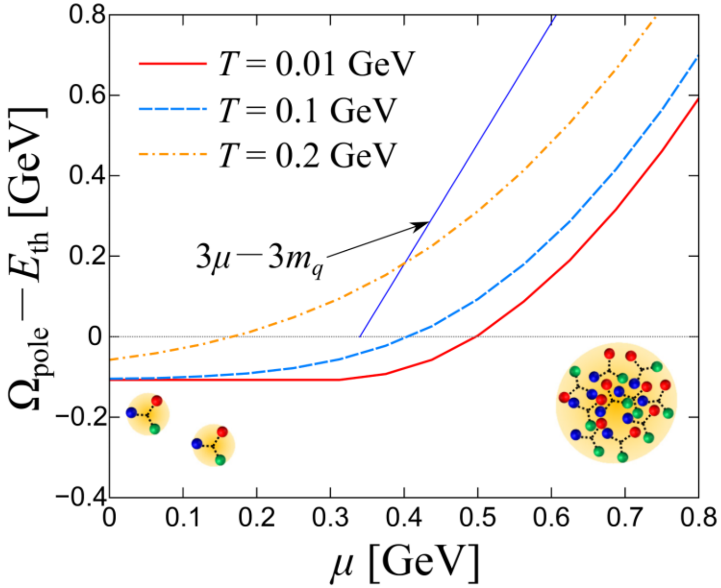
$\boldsymbol{E_B^M/E_B \simeq 0.04}$

The three-body pole survives even at high density

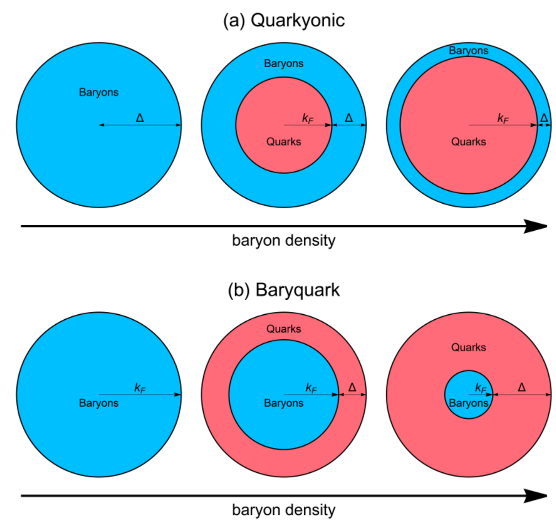
Toy model for hadron-quark crossover

HT, S. Tsutsui, T. M. Doi, and K. Iida, Symmetry **15**, 333 (2023).

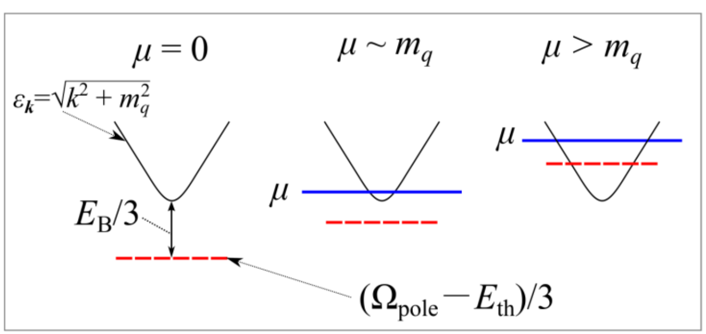
$$H=\sum_{p}\sum_{j}\varepsilon_{p}\psi^{\dagger}_{p,j}\psi_{p,j}+\sum_{k,q,k',q',P}V_{k,q,k',q',P}\psi^{\dagger}_{k,\mathrm{r}}\psi^{\dagger}_{q,\mathrm{g}}\psi^{\dagger}_{P-k-q,\mathrm{b}}\psi_{P-k'-q',\mathrm{b}}\psi_{q',\mathrm{g}}\psi_{k',\mathrm{r}}$$



Quarkyonic or Baryquark?

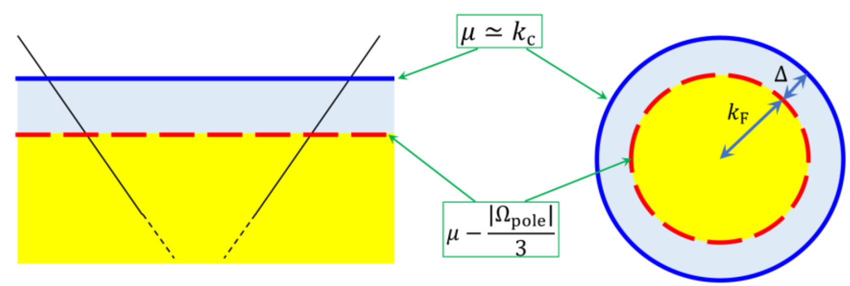


arXiv:2211.14674



$m_q=0.34$ GeV, $M_B=0.91$ GeV

Our scenario is close to quarkyonic



Non-relativistic trace anomaly

<u>Trace anomaly equation</u>

$$2\hat{H} - \hat{T}_{xx} = -\frac{g_3^2}{\sqrt{3}\pi}(\psi_\mathrm{r}^\dagger\psi_\mathrm{r})(\psi_\mathrm{g}^\dagger\psi_\mathrm{g})(\psi_\mathrm{b}^\dagger\psi_\mathrm{b})$$

$\hat{T}_{ij}$: energy-momentum tensor

W. S. Dasa, et al., Mod. Phys. Lett. A **34**, 1950291 (2019).

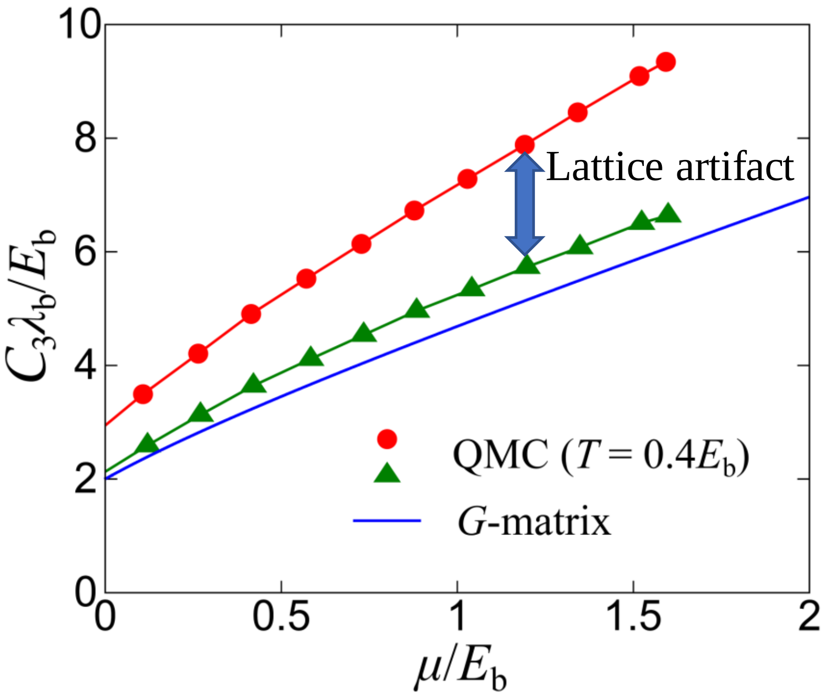
<u>Three-body contact</u>

Statistical average: $2E - P = C_3$

$$C_3 = \frac{8\sqrt{3}}{3\pi}\rho E_\mathrm{F}\frac{3E_\mathrm{F}/E_\mathrm{b}}{\left(1+\frac{3E_\mathrm{F}}{E_\mathrm{b}}\right)\left[\ln\left(1+\frac{3E_\mathrm{F}}{E_\mathrm{b}}\right)\right]^2}$$

E: energy density $\quad$ P: pressure

<u>Comparison with QMC results</u>

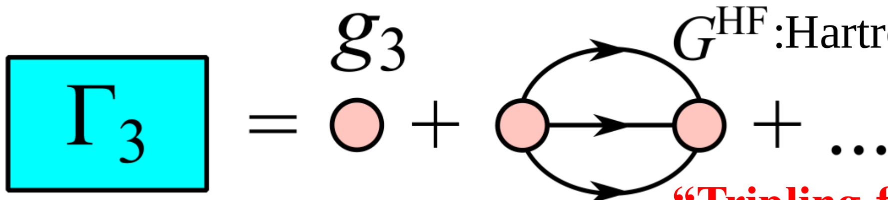


$\lambda_\mathrm{b} = \sqrt{2\pi/mE_\mathrm{b}}$: length scale associated eith E_b

QMC: J. McKenny, *et al.*, PRA **102**, 023313 (2020).

Nozières-Schmitt-Rink-type approach for the three-body crossover

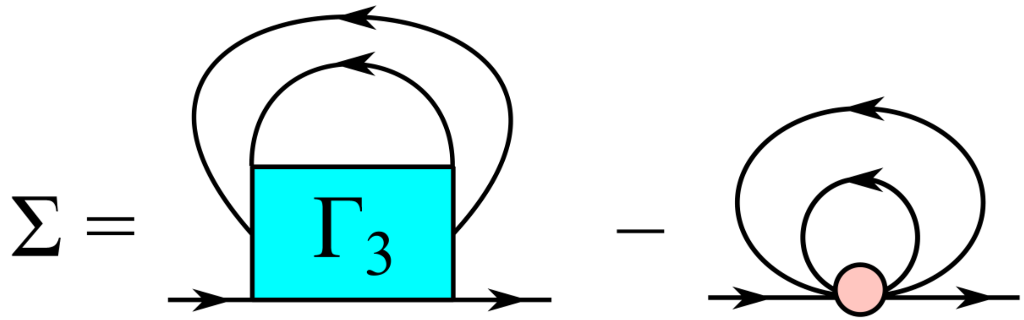
In-medium three-body *T*-matrix



G^{HF} :Hartree-Fock propagator

"Tripling fluctuations"

Self-energy for tripling fluctuations



Dyson equation

$$G_k(i\omega_\ell) = G_k^{\mathrm{HF}}(i\omega_\ell) + G_k^{\mathrm{HF}}(i\omega_\ell)\Sigma_k(i\omega_\ell)G_k(i\omega_\ell)$$

$$\simeq G_k^{\mathrm{HF}}(i\omega_\ell) + \left[G_k^{\mathrm{HF}}(i\omega_\ell)\right]^2 \Sigma_k(i\omega_\ell)$$

Truncated at O(Σ) (NSR approximation) PPNP **111**, 103739 (2020).