Jan. 12-16, 2025

The 2nd Nuclear Physics Tortoise Lecture Series (NPTLS) 2026

Pairing in ultracold atoms and neutron star

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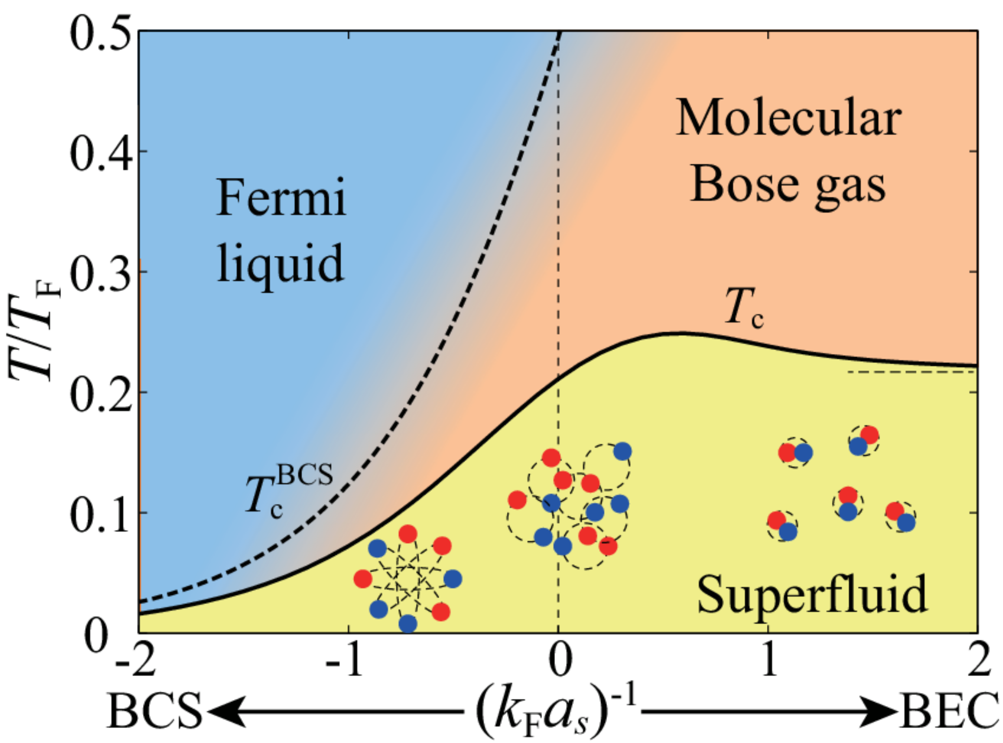
Part 7-
Exercise 2: Minimal framework for critical-temperature curve-



NSR theory

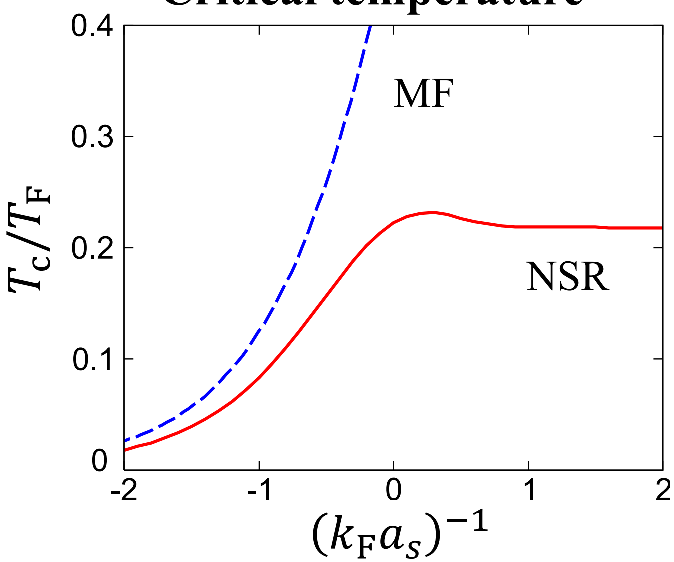
Particle number equation

$$\rho_{\mathrm{NSR}} = 2\sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) - \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)\frac{\partial \Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}{\partial \mu} e^{i\nu_\ell \delta}$$

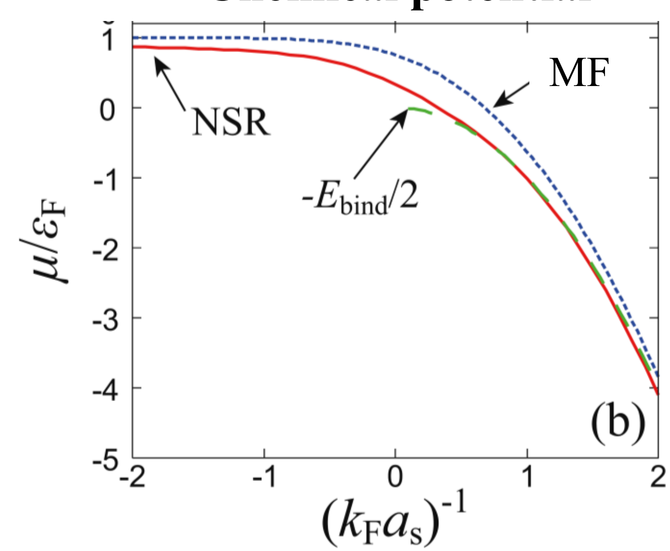
Thouless criterion

$$[T_{\mathrm{mb}}(\boldsymbol{P}=\boldsymbol{0}, i\nu_\ell = 0)]^{-1} = \frac{m}{4\pi a_s} + \sum_{\boldsymbol{k}}\left[\frac{1}{2\xi_{\boldsymbol{k}}}\tanh\left(\frac{\xi_{\boldsymbol{k}}}{2T_{\mathrm{c}}}\right) - \frac{m}{k^2}\right] = 0$$

Critical temperature



Chemical potential



NSR theory

Particle number equation

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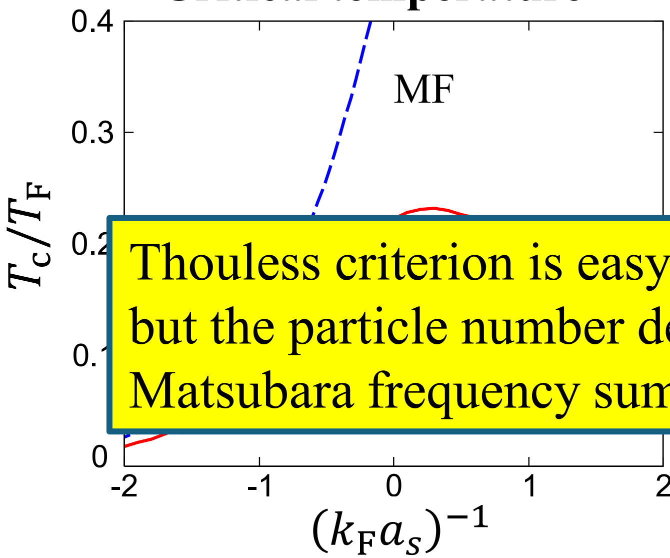
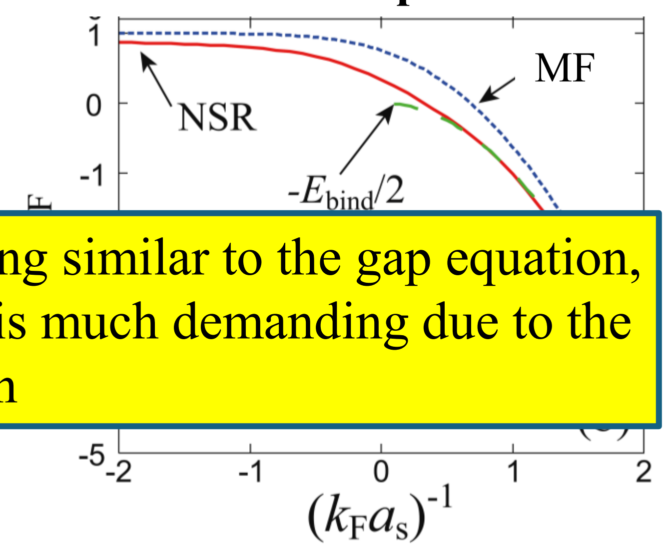


Chemical potential

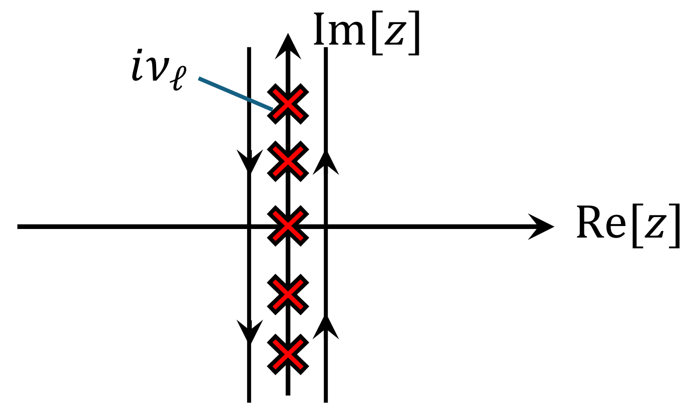


Thouless criterion is easy as being similar to the gap equation, but the particle number density is much demanding due to the Matsubara frequency summation

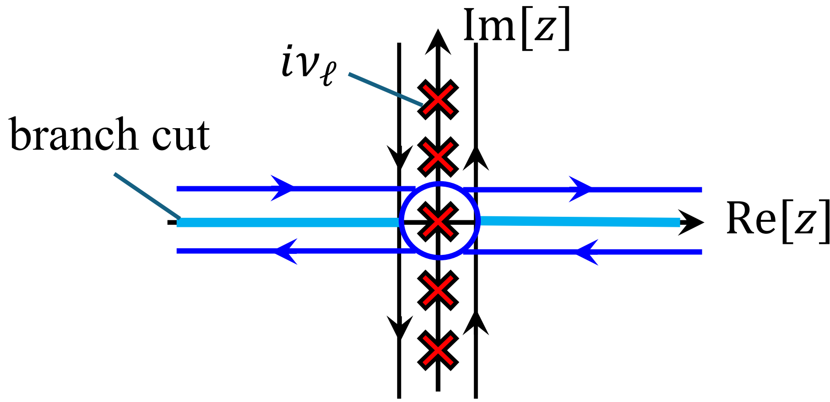
Phase shift representation

$$\Omega_{\rm NSR} - \Omega_0 = \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} \ln[1 + g\Pi_{\rm mb}(\boldsymbol{P}, i\nu_\ell)]$$

$$= \sum_{\boldsymbol{P}} \oint_C \frac{b(z)dz}{2\pi i} \ln[1 + g\Pi_{\rm mb}(\boldsymbol{P}, z)]$$



Phase shift representation



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$$= \sum_{\boldsymbol{P}} \oint_C \frac{b(z)dz}{2\pi i} \ln[1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, z)]$$

$$= \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} \frac{b(\omega)d\omega}{2\pi i} \{\ln[1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, \omega + i\delta)] - \ln[1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, \omega - i\delta)]\}$$

$$= \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} \frac{b(\omega)d\omega}{\pi} \mathrm{Im}\{\ln[1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, \omega + i\delta)]\}$$

Phase shift representation

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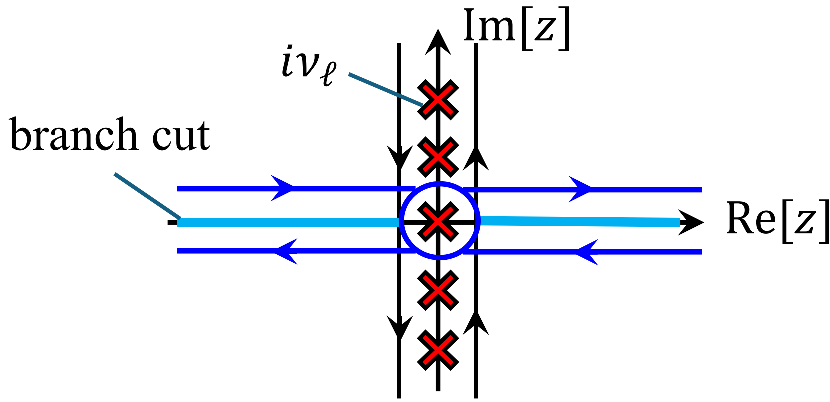
$$= \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} \frac{b(\omega)d\omega}{\pi} \mathrm{Im}\{\ln[1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, \omega + i\delta)]\}$$

$$\equiv -\sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} b(\omega)\delta_{\boldsymbol{P}}(\omega)$$



Two-particle Green's function

$$G_2(\boldsymbol{P}, \omega + i\delta) = \frac{\Pi_{\mathrm{mb}}(\boldsymbol{P}, \omega + i\delta)}{1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, \omega + i\delta)}$$

Phase shift

$$e^{i\delta_{\boldsymbol{P}}(\omega)} = \frac{G_2(\boldsymbol{P}, \omega + i\delta)}{\Pi_{\mathrm{mb}}(\boldsymbol{P}, \omega + i\delta)}$$

➡ $\delta_{\boldsymbol{P}}(\omega) = -\mathrm{Im}\ln[1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, \omega + i\delta)]$

Bosonic approximation

L. Dell'Anna and S. Grava, Condensed Matter, **6**, 16 (2021).

$$\rho_{\mathrm{NSR}} = -\frac{\partial \Omega_{\mathrm{NSR}}}{\partial \mu} = 2\sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) + \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} b(\omega) \frac{\partial}{\partial \mu} \delta_{\boldsymbol{P}}(\omega)$$

$$\frac{\partial}{\partial \mu} \delta_{\boldsymbol{P}}(\omega) \simeq \pi \left(\omega - \frac{P^2}{4m} + 2\mu - E_{\mathrm{b}}' \right) \quad + \cancel{\text{continuum}}$$

E_{b}': in-medium two-body binding energy

At $T = T_{\mathrm{c}}$, assuming the gapless condition (Thouless criterion) of bosons ($2\mu = E_{\mathrm{b}}'$),

$$\rho_{\mathrm{NSR}} \simeq 2\sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) + 2\sum_{\boldsymbol{P}} b\left(\frac{P^2}{4m}\right) = 2\sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) + 2\left(\frac{mT_{\mathrm{c}}}{\pi}\right)^{\frac{3}{2}} \zeta(3/2)$$

Bosonic approximation

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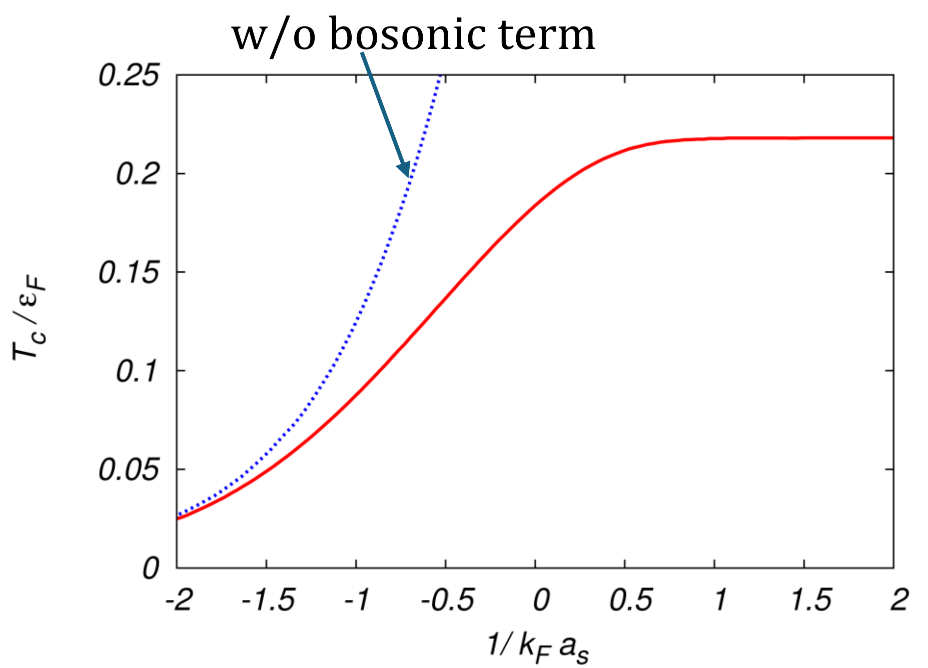
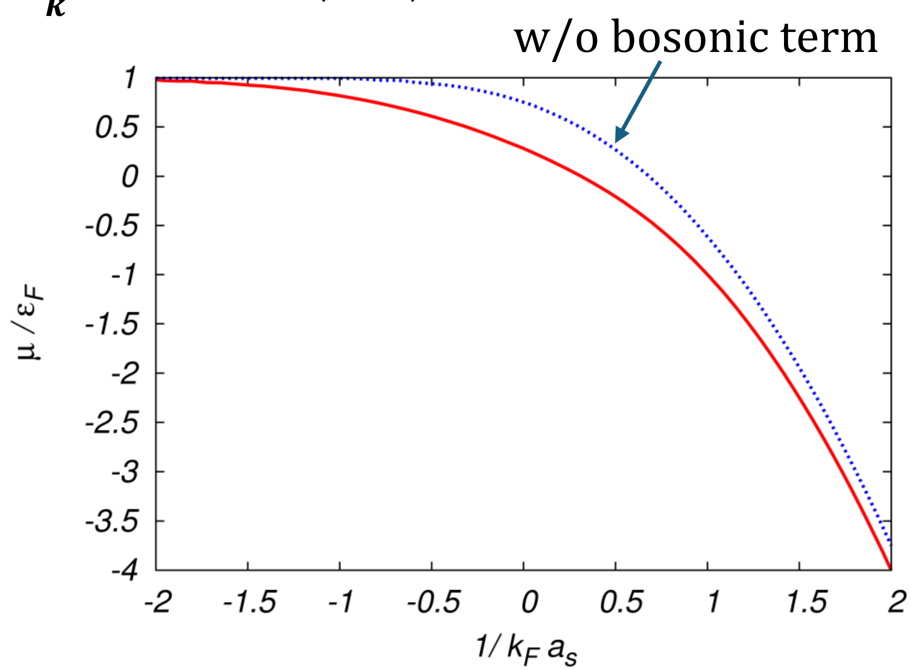
Thouless criterion: $$\frac{m}{4\pi a_s} + \sum_{\boldsymbol{k}} \left[\frac{1}{2\xi_{\boldsymbol{k}}} \tanh\left(\frac{\xi_{\boldsymbol{k}}}{2T_c} \right) - \frac{m}{k^2} \right] = 0$$

Particle number equation: $$\rho_{\text{bosonic}} = 2 \sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) + 2 \left(\frac{mT_c}{\pi} \right)^{\frac{3}{2}} \zeta(3/2)$$





Nondimensional equations

$$\frac{m}{4\pi a_s} + \sum_{\boldsymbol{k}} \left[\frac{1}{2\xi_{\boldsymbol{k}}} \tanh\left(\frac{\xi_{\boldsymbol{k}}}{2T_{\mathrm{c}}} \right) - \frac{m}{k^2} \right] = 0$$

$$\rho \equiv \rho_{\mathrm{bosonic}} = 2 \sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) + 2 \left(\frac{mT_{\mathrm{c}}}{\pi} \right)^{\frac{3}{2}} \zeta(3/2)$$

$$\frac{\pi}{2k_{\mathrm{F}} a_s} = -\int_0^{\Lambda/k_{\mathrm{F}}} \tilde{k}^2 d\tilde{k} \left[\frac{1}{\xi_{\boldsymbol{k}}/\varepsilon_{\mathrm{F}}} \tanh\left(\frac{\tilde{k}^2 - \mu/\varepsilon_{\mathrm{F}}}{2T_{\mathrm{c}}/\varepsilon_{\mathrm{F}}} \right) - \frac{1}{\tilde{k}^2} \right]$$

$$1 = 3 \int_0^{\infty} k^2 dk \, \frac{1}{e^{(\tilde{k}^2 - \mu/\varepsilon_{\mathrm{F}})/(T_{\mathrm{c}}/T_{\mathrm{F}})} + 1} + 3 \int_0^{\infty} \tilde{P}^2 d\tilde{P} \frac{1}{e^{\frac{\tilde{P}^2}{2(T_{\mathrm{c}}/\varepsilon_{\mathrm{F}})}} - 1}$$

We can numerically solve them with respect to μ/ε_F and T_c/T_F

Further extensions

BEC-BCS crossover of neutron-proton pairing in symmetric nuclear matter

➡ You can replace the Thouless criterion with that of the finite-range $^3S_1$ channel

BEC-BCS crossover of proton superconductivity in asymmetric nuclear matter

➡ You can replace the Thouless criterion with that of the finite-range $^1S_0$ channel + neutron-density-mediated interaction (new work!)