Jan. 12-16, 2025

The 2nd Nuclear Physics Tortoise Lecture Series (NPTLS) 2026

Pairing in ultracold atoms and neutron star

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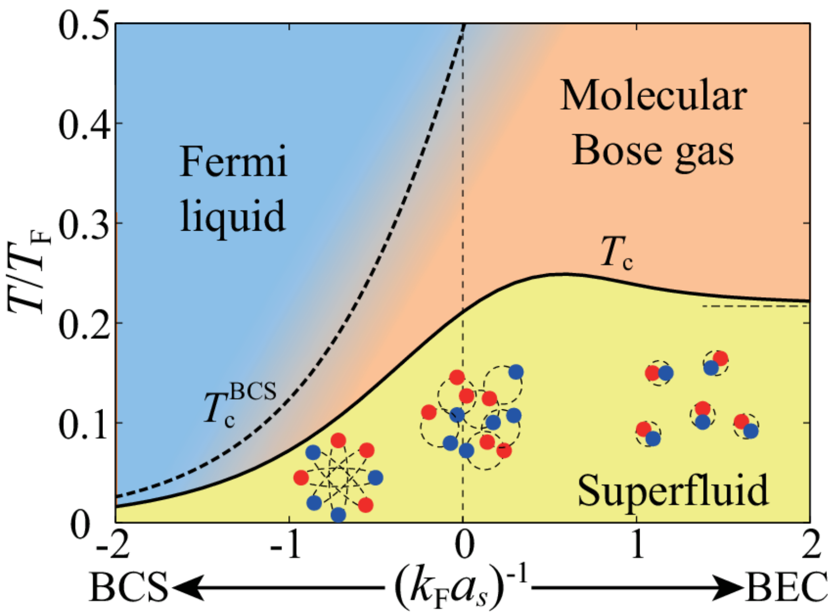
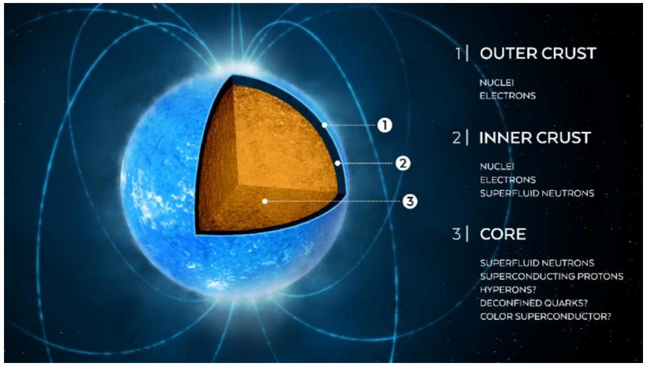
The University of Tokyo, Japan

Part 6
-Exotic superfluid and nucleon superfluid-

- Toward neutron star physics from ultracold atoms



Nucleon superfluid



A. L. Watts, *et al*., RMP **88**, 021001 (2016).

➡ Let's try to examine nucleon superfluid from an ultracold atom perspective

Outline

- Difference between a ultracold Fermi gas and neutron matter
- Neutron superfluid
- Neutron-proton pairing and proton superconductivity
- Short summary

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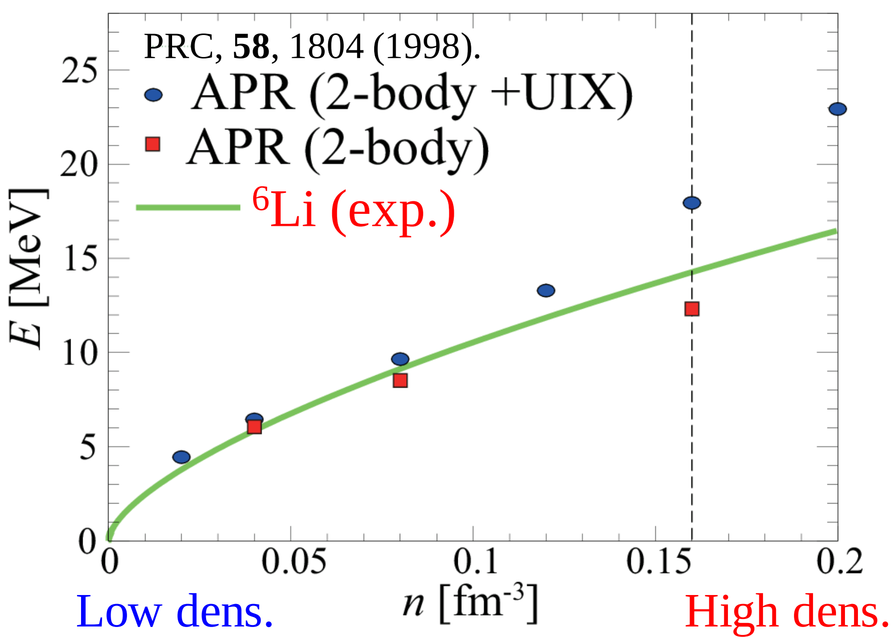
Similarities and differences

Neutron star and cold atom EOS



Similarities:

Many fermions

Strong short-range *S*-wave interaction

…

Differences:

Internal degrees of freedom (e.g., isospin)

Finite effective range

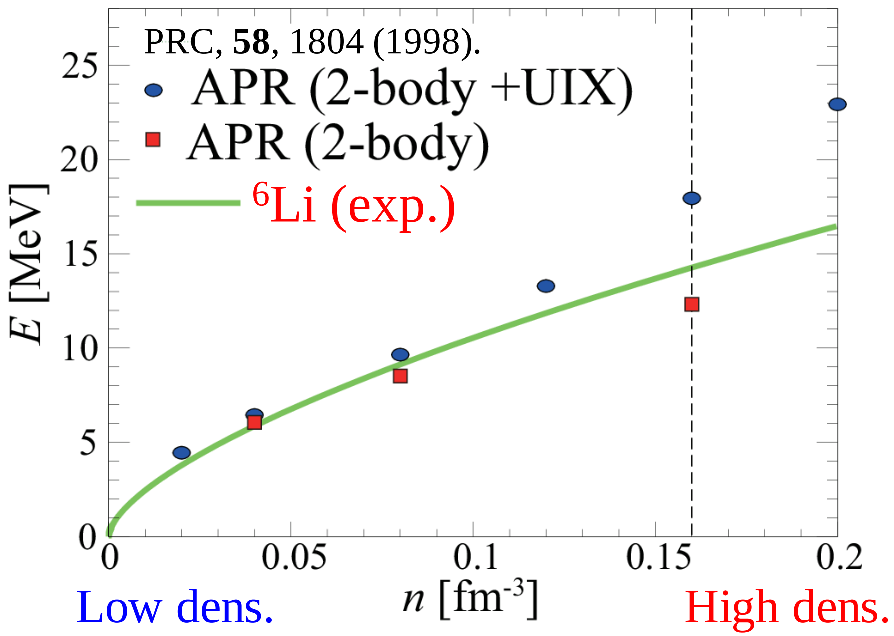
Higher-partial waves

Three-body force

…

Similarities and differences

Neutron star and cold atom EOS



Similarities:

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Differences:

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Three-body force

…

Phase shift (effective range expansion):

$$k\cot\delta_k = -\frac{1}{a_s} + \frac{1}{2}k^2 r_{\mathrm{eff}}$$

a_s: s-wave scattering length r_{eff}: effective range

r_{eff} is smaller than $|a_s|$, but non-negligible when it is comparable with $d \sim k_{\mathrm{F}}^{-1}$

Finite-range interaction

Contact-type interaction

$$\hat{V} = \frac{g}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P},s_z,s_z'} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2 s_z} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2 s_z'} c_{-\boldsymbol{k}'+\boldsymbol{P}/2 s_z'} c_{\boldsymbol{k}'+\boldsymbol{P}/2 s_z}$$

*$s_z = \pm 1/2$: neutron spin

General two-body interaction

$$\hat{V} = \frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P},s_z,s_z'} V(\boldsymbol{k},\boldsymbol{k}') c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2 s_z} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2 s_z'} c_{-\boldsymbol{k}'+\boldsymbol{P}/2 s_z'} c_{\boldsymbol{k}'+\boldsymbol{P}/2 s_z}$$

*$V(\boldsymbol{k},\boldsymbol{k}')$ does not depend on the CoM momentum $\boldsymbol{P}$

Partial-wave decomposition

$$V(\boldsymbol{k},\boldsymbol{k}') = 4\pi \sum_{\ell,m} V_\ell(k,k') Y_{\ell,m}(\hat{\boldsymbol{k}}) Y^*_{\ell,m}(\hat{\boldsymbol{k}}')$$

$Y_{\ell,m}(\hat{\boldsymbol{k}})$: spherical harmonics with $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$

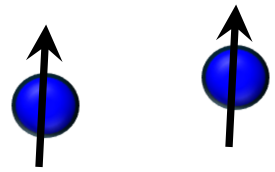
Finite-range interaction

General two-body interaction

$$\hat{V} = 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P},s_z,s_z'} \sum_{\ell,m} V_\ell(k,k') Y_{\ell,m}(\hat{\boldsymbol{k}}) Y^*_{\ell,m}(\hat{\boldsymbol{k}}')\, c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2 s_z} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2 s_z'} c_{-\boldsymbol{k}'+\boldsymbol{P}/2 s_z'} c_{\boldsymbol{k}'+\boldsymbol{P}/2 s_z}$$

Coupling of two spins

$s_z = \pm 1/2$ $\quad s_z' = \pm 1/2$



$$|s, s_z; s', s_z'\rangle = \sum_{S,S_z} |S, S_z\rangle \langle S, S_z | s, s_z; s', s_z'\rangle$$

Clebsch-Gordan coefficient

$$\hat{V} = 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P},s_z,s_z'} \sum_{S,S_z} \sum_{\ell,m} V_\ell(k,k') Y_{\ell,m}(\hat{\boldsymbol{k}}) Y^*_{\ell,m}(\hat{\boldsymbol{k}}')\, |\langle S, S_z | s, s_z; s', s_z'\rangle|^2$$

$$\times\, c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2 s_z} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2 s_z'} c_{-\boldsymbol{k}'+\boldsymbol{P}/2 s_z'} c_{\boldsymbol{k}'+\boldsymbol{P}/2 s_z}$$

*tensor force is neglected for simplicity

Finite-range interaction

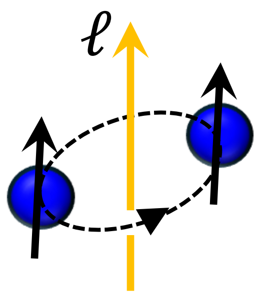
General two-body interaction with pair spin *S*

$$\hat{V} = 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P},s_z,s_z'} \sum_{S,S_z} \sum_{\ell,m} V_\ell(k,k') Y_{\ell,m}(\widehat{\boldsymbol{k}}) Y^*_{\ell,m}(\widehat{\boldsymbol{k}'}) \, |\langle S,S_z|s,s_z;s',s_z'\rangle|^2$$

$$\times\, c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2 s_z} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2 s_z'} c_{-\boldsymbol{k}'+\boldsymbol{P}/2 s_z'} c_{\boldsymbol{k}'+\boldsymbol{P}/2 s_z}$$

Coupling of pair spin and angular momentum

$$|\ell,m;S,S_z\rangle = \sum_{J,J_z} |J,J_z\rangle\langle J,J_z|\ell,m;S,S_z\rangle$$

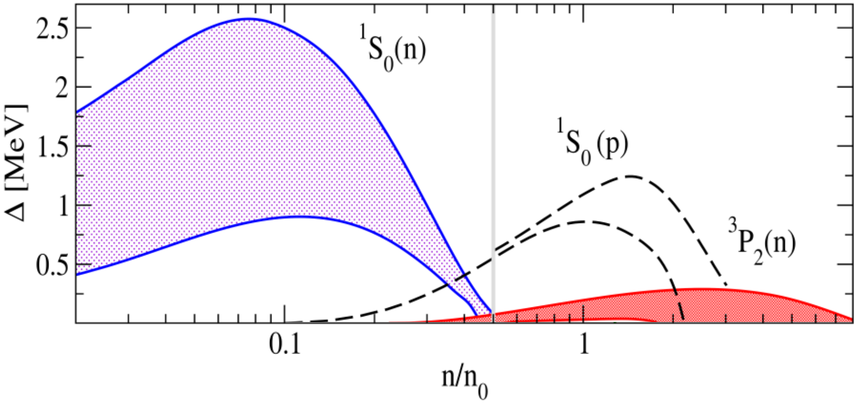


$$\hat{V} = 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \sum_{J,J_z} \sum_{S,S_z} \sum_{\ell,m} \sum_{s_z,s_z'} V_\ell(k,k') |\langle J,J_z|\ell,m;S,S_z\rangle|^2 |\langle S,S_z|s,s_z;s',s_z'\rangle|^2$$

$$\times\, Y_{\ell,m}(\widehat{\boldsymbol{k}}) Y^*_{\ell,m}(\widehat{\boldsymbol{k}'}) c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2 s_z} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2 s_z'} c_{-\boldsymbol{k}'+\boldsymbol{P}/2 s_z'} c_{\boldsymbol{k}'+\boldsymbol{P}/2 s_z}$$

Example of neutron-neutron pairing

Nucleon pairing channel



Eur. Phys. J. A (2019) **55**: 167

Term symbol: $^{2S+1}\ell_J$

$^1S_0\ (S=0, \ell=0, J=0)$

$^3P_2\ (S=1, \ell=1, J=2)$

$$\hat{V}_{^1S_0} = 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \sum_{s_z, s_z'} V_0(k,k')|\langle 0,0|0,0;0,0\rangle|^2 |\langle 0,0|1/2,s_z;1/2,s_z'\rangle|^2$$

$$\times Y_{0,0}(\hat{\boldsymbol{k}}) Y_{0,0}^*(\hat{\boldsymbol{k}}') c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2 s_z} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2 s_z'} c_{-\boldsymbol{k}'+\boldsymbol{P}/2 s_z'} c_{\boldsymbol{k}'+\boldsymbol{P}/2 s_z}$$

$$= \frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \sum_{s_z, s_z'} V_0(k,k') c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2 s_z} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2 s_z'} c_{-\boldsymbol{k}'+\boldsymbol{P}/2 s_z'} c_{\boldsymbol{k}'+\boldsymbol{P}/2 s_z}$$

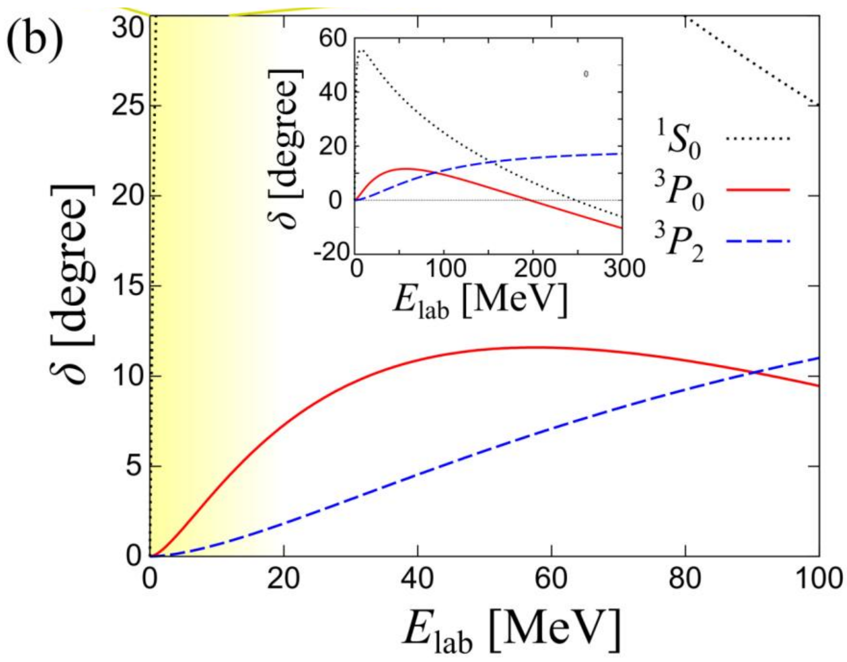
Example of neutron-neutron pairing

$$
\begin{aligned}
V_{^3P_2} &= 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \sum_{m} \sum_{S_z} \sum_{s_z,s_z'} V_1(k,k') Y_{1,m}(\hat{\boldsymbol{k}}) Y^*_{1,m}(\hat{\boldsymbol{k}}') \langle 1,m;1,S_z|2,J_z\rangle^2 \langle s,s_z;s,s_z|1,S_z\rangle^2 \\
&\quad \times c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,s_z} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,s_z'} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,s_z'} c_{\boldsymbol{k}'+\boldsymbol{P}/2,s_z} \\
&= 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} V_1(k,k') \left[Y_{1,1}(\hat{\boldsymbol{k}}) Y^*_{1,1}(\hat{\boldsymbol{k}}') + \frac{1}{2} Y_{1,0}(\hat{\boldsymbol{k}}) Y^*_{1,0}(\hat{\boldsymbol{k}}') + \frac{1}{6} Y_{1,-1}(\hat{\boldsymbol{k}}) Y^*_{1,-1}(\hat{\boldsymbol{k}}') \right] \\
&\quad \times c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,1/2} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,1/2} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,1/2} c_{\boldsymbol{k}'+\boldsymbol{P}/2,1/2} \quad (s_z = s_z' = 1/2) \\
&+ 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} V_1(k,k') \left[\frac{1}{6} Y_{1,1}(\hat{\boldsymbol{k}}) Y^*_{1,1}(\hat{\boldsymbol{k}}') + \frac{1}{2} Y_{1,0}(\hat{\boldsymbol{k}}) Y^*_{1,0}(\hat{\boldsymbol{k}}') + Y_{1,-1}(\hat{\boldsymbol{k}}) Y^*_{1,-1}(\hat{\boldsymbol{k}}') \right] \\
&\quad \times c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,-1/2} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,-1/2} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,-1/2} c_{\boldsymbol{k}'+\boldsymbol{P}/2,-1/2} \quad (s_z = s_z' = -1/2) \\
&+ 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} V_1(k,k') \left[\frac{1}{2} Y_{1,1}(\hat{\boldsymbol{k}}) Y^*_{1,1}(\hat{\boldsymbol{k}}') + \frac{2}{3} Y_{1,0}(\hat{\boldsymbol{k}}) Y^*_{1,0}(\hat{\boldsymbol{k}}') + \frac{1}{2} Y_{1,-1}(\hat{\boldsymbol{k}}) Y^*_{1,-1}(\hat{\boldsymbol{k}}') \right] \\
&\quad \times c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,1/2} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,-1/2} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,-1/2} c_{\boldsymbol{k}'+\boldsymbol{P}/2,1/2} \quad (s_z = 1/2, \quad s_z' = -1/2) \\
&+ 2\pi \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} V_1(k,k') \left[\frac{1}{2} Y_{1,1}(\hat{\boldsymbol{k}}) Y^*_{1,1}(\hat{\boldsymbol{k}}') + \frac{2}{3} Y_{1,0}(\hat{\boldsymbol{k}}) Y^*_{1,0}(\hat{\boldsymbol{k}}') + \frac{1}{2} Y_{1,-1}(\hat{\boldsymbol{k}}) Y^*_{1,-1}(\hat{\boldsymbol{k}}') \right] \\
&\quad \times c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,-1/2} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,1/2} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,1/2} c_{\boldsymbol{k}'+\boldsymbol{P}/2,-1/2} \quad (s_z = -1/2, \quad s_z' = 1/2).
\end{aligned}
$$

Other pairing channels?

HT, H. Funaki, Y. Sekino, N. Yasutake, and M. Matsuo, Phys. Rev. C **108**, L052802 (2023).

$^3P_0$ channel is also attractive
Moreover stronger than $^3P_2$ at low energies



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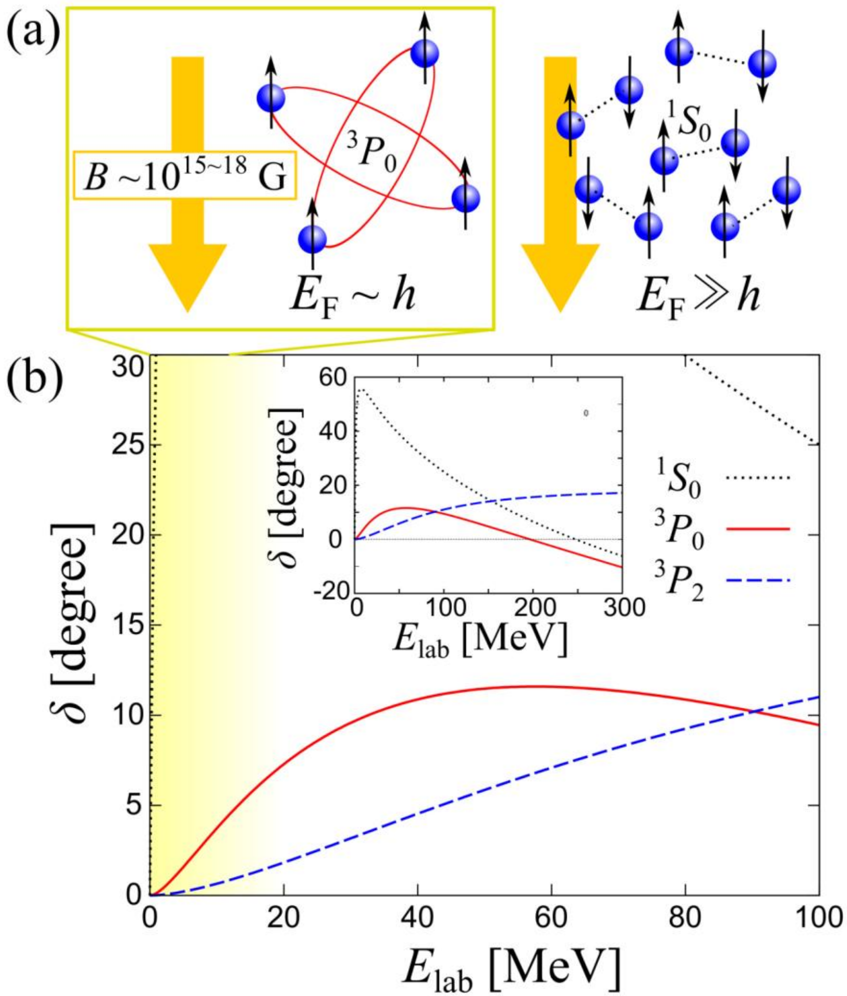
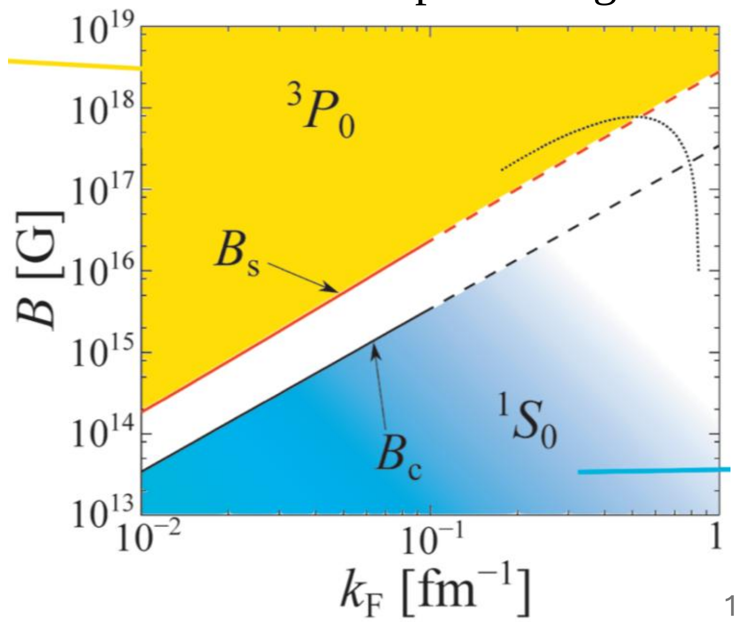


$^3P_0$ channel is also attractive
Moreover stronger than $^3P_2$ at low energies
➡ In the spin-polarized case, $^3P_0$ superfluid may appear in the dilute regime due to the suppression of $^1S_0$ pairing

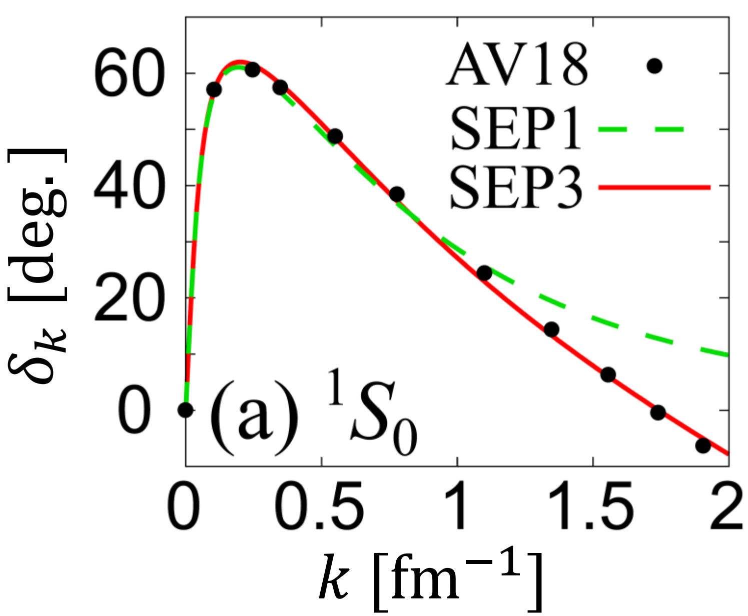
Ground-state phase diagram



$^{1}S_{0}$ channel interaction

$$\hat{V}_{^1S_0} = \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} V_0(k,k') c^{\dagger}_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow} c^{\dagger}_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow} c_{-\boldsymbol{k}'+\boldsymbol{P}/2\downarrow} c_{\boldsymbol{k}'+\boldsymbol{P}/2\uparrow}$$

We need to determine $V_0(k,k')$. How can we do that?



HT, T Hatsuda, P van Wyk, Y Ohashi
Sc. Rep. **9**, 18477 (2019).

➡ **Reproduce the phase shift**

$$k\cot\delta_k = -\frac{1}{a_s} + \frac{1}{2}k^2 r_{\mathrm{eff}}$$

Once the phase shift is reproduced, two-body physics does not depend on the construction of the interactions
(regardless of e.g., chiral EFT, $V_{\mathrm{low}\,k}, \ldots$)

Separable Yamaguchi potential

Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

$$V_0(k,k') = g\gamma_k\gamma_{k'}$$

Form factor: $\gamma_k = \dfrac{1}{1+(k/\Lambda)^2}$

*Simplification for offshell component, but useful because of analytical form of *T* matrix

Two-body *T* matrix

$$T(\boldsymbol{k},\boldsymbol{k}';\omega) = V_0(k,k') + \sum_{\boldsymbol{p}} V_0(k,p)\frac{1}{\omega+i\delta-p^2/m}T(\boldsymbol{p},\boldsymbol{k}';\omega)$$

$$= g\gamma_k\gamma_{k'} + g\gamma_k\sum_{\boldsymbol{p}}\frac{\gamma_p^2}{\omega+i\delta-p^2/m}\gamma_{k'} + \cdots$$

All terms are proportional to $\gamma_k\gamma_{k'}$

$$\Rightarrow T(\boldsymbol{k},\boldsymbol{k}';\omega) = \gamma_k\mathcal{T}(\omega)\gamma_{k'}$$

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Two-body *T* matrix

$$T(\boldsymbol{k},\boldsymbol{k}';\omega) = V_0(k,k') + \sum_{\boldsymbol{p}} V_0(k,p)\frac{1}{\omega + i\delta - p^2/m}T(\boldsymbol{p},\boldsymbol{k}';\omega)$$

$$\equiv \gamma_k\mathcal{T}(\omega)\gamma_{k'} = g\gamma_k\gamma_{k'} + g\gamma_k\gamma_{k'}\sum_{\boldsymbol{p}}\frac{\gamma_p^2}{\omega + i\delta - p^2/m}\mathcal{T}(\omega)$$

$$\boxed{\mathcal{T}(\omega) = \frac{g}{1 - g\Pi_{\mathrm{Y}}(\omega)} \qquad \Pi_{\mathrm{Y}}(\omega) = \sum_{\boldsymbol{p}}\frac{\gamma_p^2}{\omega + i\delta - p^2/m} = \frac{m\Lambda^3}{8\pi}\frac{1}{\left(\sqrt{m\omega} + i\Lambda\right)^2}}$$

Separable Yamaguchi potential

Phase shift of Yamaguchi potential

$$[T(\boldsymbol{k},\boldsymbol{k};2\varepsilon_{\boldsymbol{k}})]^{-1} = \left[1+\left(\frac{k}{\Lambda}\right)^2\right]^2\left[\frac{1}{g}-\frac{m\Lambda^3}{8\pi(k+i\Lambda)^2}\right]$$

$$= \frac{m}{4\pi}(k\cot\delta_k - ik)$$

$$\simeq \frac{m}{4\pi}\left(-\frac{1}{a_s}+\frac{1}{2}r_{\text{eff}}k^2 - ik\right)$$

NN parameter

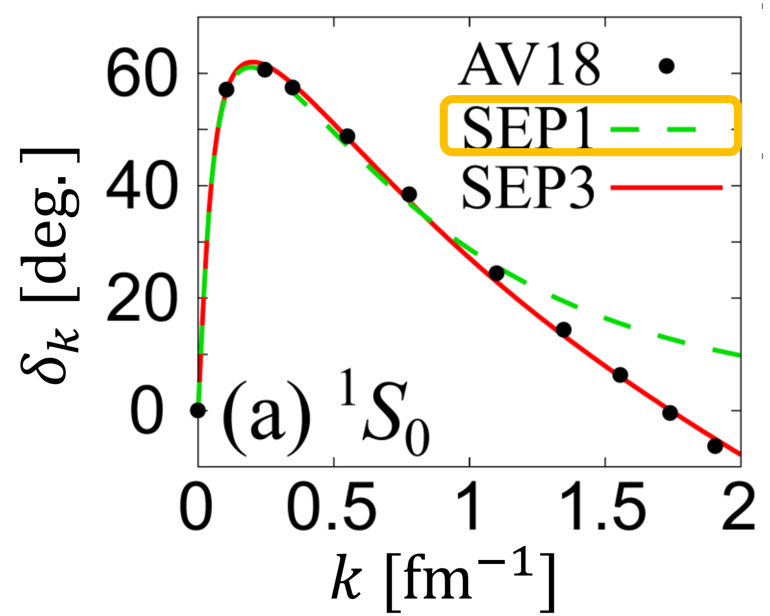
$a_s \simeq -18.5$ fm

$r_{\text{eff}} \simeq 2.75$ fm

$$g = -\frac{4\pi}{m}\left(\frac{\Lambda}{2}-\frac{1}{a_s}\right)^{-1}$$

$$\Lambda = \frac{3}{2r_{\text{eff}}}\left(1+\sqrt{1-\frac{16r_{\text{eff}}}{9a_s}}\right)$$

Reproducing $\boldsymbol{\delta_k}$ up to $\boldsymbol{k \sim 1}$ fm^{-1}



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BCS-BEC crossover with separable finite-range potential

$$H = \sum_{\boldsymbol{k}\boldsymbol{\sigma}} \xi_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} V_0(k,k') c^{\dagger}_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow} c^{\dagger}_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow} c_{-\boldsymbol{k}'+\boldsymbol{P}/2\downarrow} c_{\boldsymbol{k}'+\boldsymbol{P}/2\uparrow}$$

Pairing term

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} V_0(k,k') \langle c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} \rangle$$

Hartree term (non-negligible for finite range)

$$\Sigma_{\sigma}(\boldsymbol{k}) = \sum_{\boldsymbol{k}'} V_0\left(\frac{|\boldsymbol{k}-\boldsymbol{k}'|}{2}, \frac{|\boldsymbol{k}-\boldsymbol{k}'|}{2}\right) \langle c^{\dagger}_{\boldsymbol{k}'\bar{\sigma}} c_{\boldsymbol{k}'\bar{\sigma}} \rangle$$

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Mean-field Hamiltonian

$$H_{\mathrm{MF}} = \sum_{\boldsymbol{k}\boldsymbol{\sigma}} [\xi_{\boldsymbol{k}} + \Sigma_\sigma(\boldsymbol{k})] c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \sum_{\boldsymbol{k}} \left[\Delta(\boldsymbol{k}) c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} + \Delta^*(\boldsymbol{k}) c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow}\right]$$

$$- \sum_{\boldsymbol{k},\boldsymbol{k'}} \left[V_0(k,k') \langle c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} \rangle \langle c_{-\boldsymbol{k'}\downarrow} c_{\boldsymbol{k'}\uparrow} \rangle\right] - \sum_{\boldsymbol{p},\boldsymbol{p'}} \left[V_0\left(\frac{|\boldsymbol{p}-\boldsymbol{p'}|}{2}, \frac{|\boldsymbol{p}-\boldsymbol{p'}|}{2}\right) \langle c^\dagger_{\boldsymbol{p}\uparrow} c_{\boldsymbol{p}\uparrow} \rangle \langle c^\dagger_{\boldsymbol{p'}\downarrow} c_{\boldsymbol{p'}\downarrow} \rangle\right]$$

Hartree-Fock-Bogoliubov theory

*but Fock term is absent

Separable condition: $\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} V_0(k,k')\left\langle c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow}\right\rangle = -g\gamma_k \sum_{\boldsymbol{k}'} \gamma_{k'} \left\langle c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow}\right\rangle \equiv \Delta\gamma_k$

Spin unpolarized: $\Sigma(\boldsymbol{k}) = \Sigma_\uparrow(\boldsymbol{k}) = \Sigma_\downarrow(\boldsymbol{k}), n_{\boldsymbol{k}} = \langle c^\dagger_{\boldsymbol{k}\uparrow} c_{\boldsymbol{k}\uparrow}\rangle = \langle c^\dagger_{\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\downarrow}\rangle$

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$$\Rightarrow \quad H_{\mathrm{MF}} = \sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} \hat{H}_{\mathrm{BdG}}(\boldsymbol{k}) \Psi_{\boldsymbol{k}} - \frac{|\Delta|^2}{g} + \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}} - \sum_{\boldsymbol{p}} [\Sigma(\boldsymbol{k}) n_{\boldsymbol{k}}]$$

$$\hat{H}_{\mathrm{BdG}}(\boldsymbol{k}) = \begin{pmatrix} \xi_{\boldsymbol{k}} + \Sigma(\boldsymbol{k}) & -\Delta\gamma_k \\ -\Delta^*\gamma_k & -\xi_{\boldsymbol{k}} - \Sigma(\boldsymbol{k}) \end{pmatrix}$$

Eigenenergy:

$$\pm E_{\boldsymbol{k}} = \pm\sqrt{\{\xi_{\boldsymbol{k}} + \Sigma(\boldsymbol{k})\}^2 + |\Delta(\boldsymbol{k})|^2}$$

Hartree-Fock-Bogoliubov theory

*but Fock term is absent

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$$H_{\mathrm{MF}} = \sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} \hat{H}_{\mathrm{BdG}}(\boldsymbol{k})\Psi_{\boldsymbol{k}} - \frac{|\Delta|^2}{g} + \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}} - \sum_{\boldsymbol{p}} [\Sigma(\boldsymbol{k}) n_{\boldsymbol{k}}]$$

$$\hat{H}_{\mathrm{BdG}}(\boldsymbol{k}) = \begin{pmatrix} \xi_{\boldsymbol{k}} + \Sigma(\boldsymbol{k}) & -\Delta\gamma_k \\ -\Delta^*\gamma_k & -\xi_{\boldsymbol{k}} - \Sigma(\boldsymbol{k}) \end{pmatrix}$$

Eigenenergy:

$$\pm E_{\boldsymbol{k}} = \pm\sqrt{\{\xi_{\boldsymbol{k}} + \Sigma(\boldsymbol{k})\}^2 + |\Delta(\boldsymbol{k})|^2}$$

Bogoliubov transformation (same procedure with contact interaction)

Gap equation

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} V_0(k,k') \frac{\Delta(\boldsymbol{k}')}{2E_{\boldsymbol{k}}} \tanh\left(\frac{\beta E_{\boldsymbol{k}}}{2}\right)$$

Neutron number density

$$\rho = 2\sum_{\boldsymbol{k}} [|u_{\boldsymbol{k}}|^2 f(E_{\boldsymbol{k}}) + |v_{\boldsymbol{k}}|^2\{1 - f(E_{\boldsymbol{k}})\}]$$

Self-consistent equations

Gap equation

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} V_0(k, k') \frac{\Delta(\boldsymbol{k}')}{2E_{\boldsymbol{k}}} \tanh\left(\frac{\beta E_{\boldsymbol{k}}}{2}\right)$$

$$\boxed{\frac{m}{4\pi a_s} = \frac{1}{g} + \sum_{\boldsymbol{k}} \frac{m\gamma_k^2}{k^2}}$$

$$\frac{m}{4\pi a_s} = -\sum_{\boldsymbol{k}} \gamma_k^2 \left[\frac{1}{2E_{\boldsymbol{k}}} \tanh\left(\frac{\beta E_{\boldsymbol{k}}}{2}\right) - \frac{m}{k^2}\right]$$

Neutron number density

$$\rho \equiv 2\sum_{\boldsymbol{k}} n_{\boldsymbol{k}} \equiv 2\sum_{\boldsymbol{k}} [|u_{\boldsymbol{k}}|^2 f(E_{\boldsymbol{k}}) + |v_{\boldsymbol{k}}|^2\{1 - f(E_{\boldsymbol{k}})\}]$$

Self-energy equation

$$\Sigma(\boldsymbol{k}) = \sum_{\boldsymbol{p}} V_0\left(\frac{|\boldsymbol{k}-\boldsymbol{p}|}{2}, \frac{|\boldsymbol{k}-\boldsymbol{p}|}{2}\right) n_{\boldsymbol{p}} = g\sum_{\boldsymbol{p}} \gamma^2_{\frac{|\boldsymbol{p}-\boldsymbol{k}|}{2}} n_{\boldsymbol{p}}$$

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BCS

For given $\boldsymbol{\rho}, \boldsymbol{a_s}, \boldsymbol{r}_{\mathbf{eff}}$

$$\mu/\varepsilon_{\mathrm{F}} \simeq 1$$
$$\Sigma(\boldsymbol{k})/\varepsilon_{\mathrm{F}} \simeq 0$$

Gap eq. ➡ $\Delta/\varepsilon_{\mathrm{F}}$

Self-consistent equations

Gap equation

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$\mu/\varepsilon_{\mathrm{F}} \simeq 1$

$\Sigma(\boldsymbol{k})/\varepsilon_{\mathrm{F}} \simeq 0$

Gap eq. ➡ $\Delta/\varepsilon_{\mathrm{F}}$

HF-BCS

For given $\boldsymbol{\rho}, \boldsymbol{a_s}, \boldsymbol{r}_{\mathrm{eff}}$

First, HF calculation

$\mu/\varepsilon_{\mathrm{F}}$, $\Sigma(\boldsymbol{k})/\varepsilon_{\mathrm{F}}$ $(\Delta = 0)$

Then, Gap eq. ➡ $\Delta/\varepsilon_{\mathrm{F}}$

Self-consistent equations

Gap equation

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} V_0(k,k')\frac{\Delta(\boldsymbol{k}')}{2E_{\boldsymbol{k}}}\tanh\left(\frac{\beta E_{\boldsymbol{k}}}{2}\right)$$

$$\boxed{\frac{m}{4\pi a_s} = \frac{1}{g} + \sum_{\boldsymbol{k}}\frac{m\gamma_k^2}{k^2}} \Rightarrow \frac{m}{4\pi a_s} = -\sum_{\boldsymbol{k}}\gamma_k^2\left[\frac{1}{2E_{\boldsymbol{k}}}\tanh\left(\frac{\beta E_{\boldsymbol{k}}}{2}\right) - \frac{m}{k^2}\right]$$

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$$\rho \equiv 2\sum_{\boldsymbol{k}} n_{\boldsymbol{k}} \equiv 2\sum_{\boldsymbol{k}}[|u_{\boldsymbol{k}}|^2 f(E_{\boldsymbol{k}}) + |v_{\boldsymbol{k}}|^2\{1 - f(E_{\boldsymbol{k}})\}]$$

Self-energy equation

$$\Sigma(\boldsymbol{k}) = \sum_{\boldsymbol{p}} V_0\left(\frac{|\boldsymbol{k}-\boldsymbol{p}|}{2},\frac{|\boldsymbol{k}-\boldsymbol{p}|}{2}\right)n_{\boldsymbol{p}} = g\sum_{\boldsymbol{p}}\gamma^2_{\frac{|\boldsymbol{p}-\boldsymbol{k}|}{2}}n_{\boldsymbol{p}}$$

BCS

For given $\boldsymbol{\rho}, \boldsymbol{a_s}, \boldsymbol{r}_{\mathbf{eff}}$

$\mu/\varepsilon_{\mathrm{F}} \simeq 1$

$\Sigma(\boldsymbol{k})/\varepsilon_{\mathrm{F}} \simeq 0$

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HF-BCS

For given $\boldsymbol{\rho}, \boldsymbol{a_s}, \boldsymbol{r}_{\mathbf{eff}}$

First, HF calculation

$\mu/\varepsilon_{\mathrm{F}}$, $\Sigma(\boldsymbol{k})/\varepsilon_{\mathrm{F}}$ $(\Delta = 0)$

Then, Gap eq. ➡ $\Delta/\varepsilon_{\mathrm{F}}$

HFB

For given $\boldsymbol{\rho}, \boldsymbol{a_s}, \boldsymbol{r}_{\mathbf{eff}}$

obtain 3 quantities

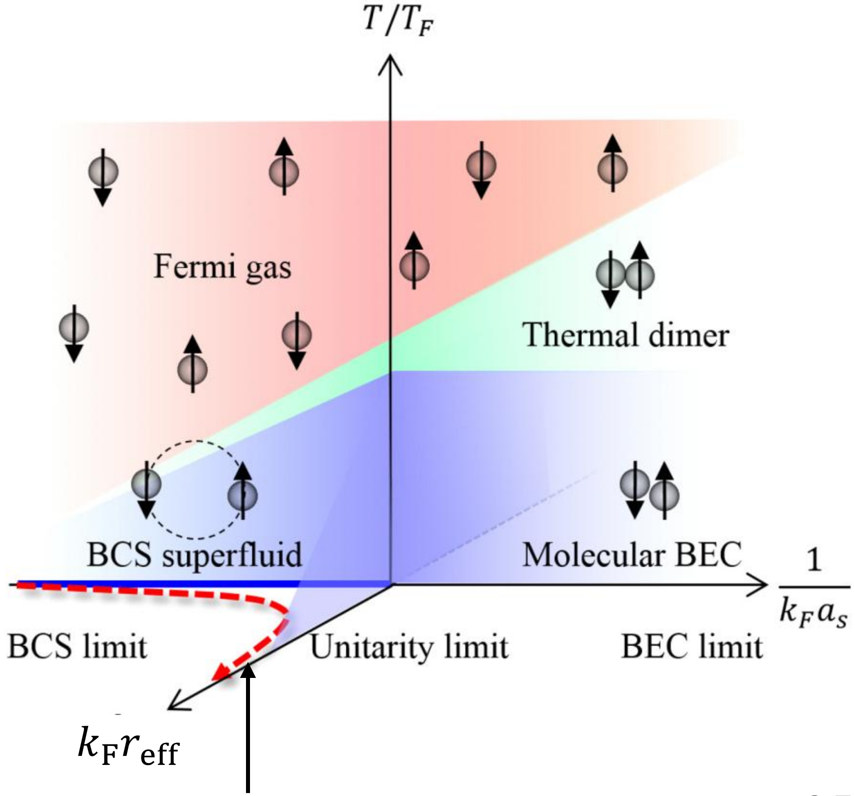
$\Delta/\varepsilon_{\mathrm{F}}$ $\mu/\varepsilon_{\mathrm{F}}$ $\Sigma(\boldsymbol{k})/\varepsilon_{\mathrm{F}}$

simultaneously

Density-induced BCS-BEC crossover

Generalization to finite range

M. Horikoshi and M. Kuwata-Gonokami,
J. Mod. Phys. E **28**, 1930001 (2019)



Neutron matter: $(k_\mathrm{F} r_\mathrm{eff})(k_\mathrm{F} a_s)^{-1} = -\frac{2.7}{18.5}$

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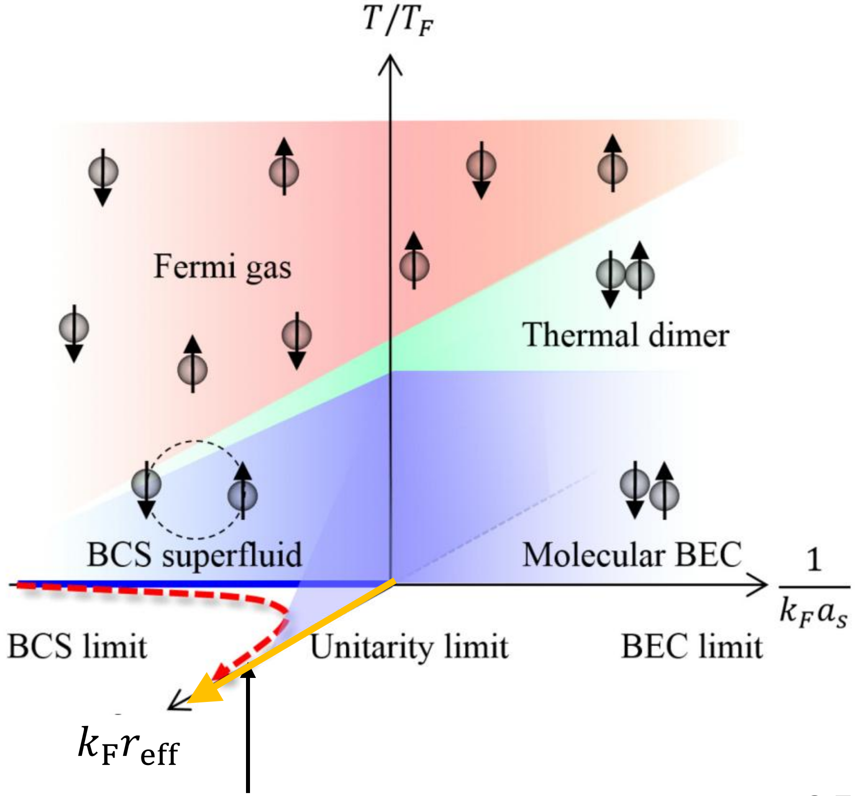
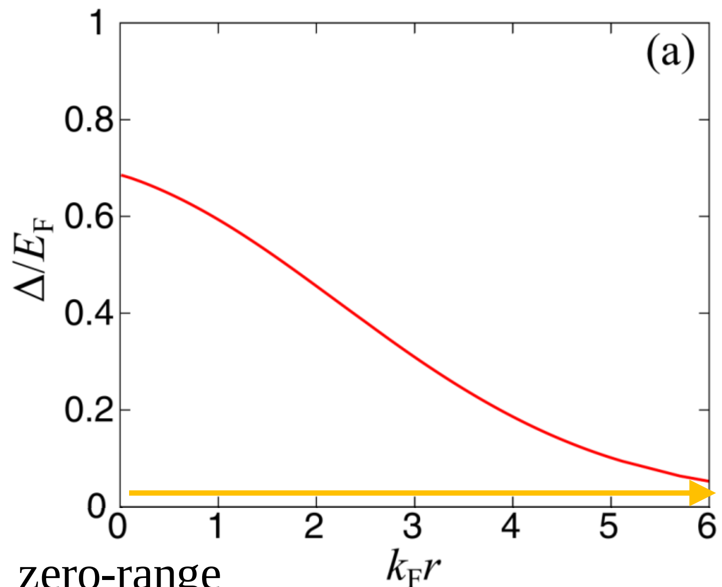


Neutron matter: $(k_\mathrm{F} r_\mathrm{eff})(k_\mathrm{F} a_s)^{-1} = -\frac{2.7}{18.5}$

HFB result at unitarity (1/*a* = 0)

HT and H. Liang, Phys. Rev. A **106**, 043308 (2022).



➡ Finite range suppresses the pairing

Pairing fluctuations with zero range

Many-body *T* matrix

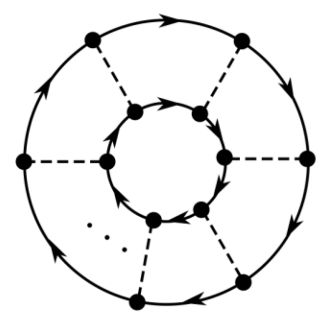
$$T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) = \frac{g}{1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}$$

Pair propagator

$$\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) = -\sum_{\boldsymbol{k}} \frac{1 - f(\xi_{\boldsymbol{k}+\boldsymbol{P}/2}) - f(\xi_{\boldsymbol{k}+\boldsymbol{P}/2})}{i\nu_\ell - \xi_{\boldsymbol{k}+\boldsymbol{P}/2} - \xi_{\boldsymbol{k}+\boldsymbol{P}/2}}$$

NSR number density with separable finite-range interaction

$$\rho_{\mathrm{NSR}} = 2\sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) - \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} [T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) \quad] \frac{\partial \Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}{\partial \mu} e^{i\nu_\ell \delta}$$



Thouless criterion

$$[T_{\mathrm{mb}}(\boldsymbol{P} = \boldsymbol{0}, i\nu_\ell = 0)]^{-1} = \frac{m}{4\pi a_s} + \sum_{\boldsymbol{k}} \left[\frac{1}{2\xi_{\boldsymbol{k}}} \tanh\left(\frac{\xi_{\boldsymbol{k}}}{2T_{\mathrm{c}}}\right) - \frac{m}{k^2}\right] = 0$$

Pairing fluctuations with finite range

Many-body *T* matrix

$$T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) = \frac{g}{1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}$$

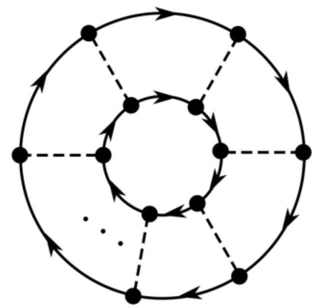
Pair propagator

$$\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) = -\sum_{\boldsymbol{k}} \gamma_k^2 \frac{1 - f(\xi^*_{\boldsymbol{k}+\boldsymbol{P}/2}) - f(\xi^*_{\boldsymbol{k}+\boldsymbol{P}/2})}{i\nu_\ell - \xi^*_{\boldsymbol{k}+\boldsymbol{P}/2} - \xi^*_{\boldsymbol{k}+\boldsymbol{P}/2}}$$

$\xi^*_{\boldsymbol{k}} = \xi_{\boldsymbol{k}} + \Sigma(\boldsymbol{k})$: Hartree-shifted dispersion

NSR number density with separable finite-range interaction

$$\rho_{\mathrm{NSR}} = 2\sum_{\boldsymbol{k}} f(\xi^*_{\boldsymbol{k}}) - \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} [T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) - g] \frac{\partial \Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}{\partial \mu} e^{i\nu_\ell \delta}$$



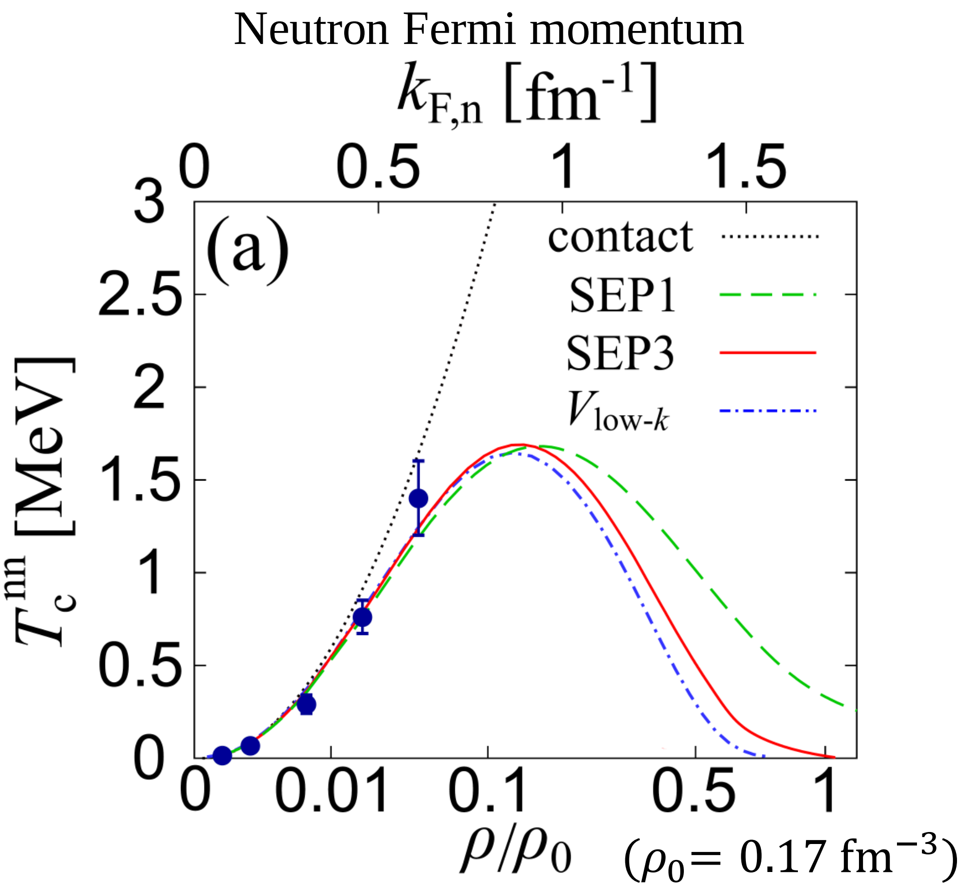
1st order (Hartree) term is subtracted

Thouless criterion

$$[T_{\mathrm{mb}}(\boldsymbol{P} = \boldsymbol{0}, i\nu_\ell = 0)]^{-1} = \frac{m}{4\pi a_s} + \sum_{\boldsymbol{k}} \gamma_k^2 \left[\frac{1}{2\xi^*_{\boldsymbol{k}}} \tanh\left(\frac{\xi^*_{\boldsymbol{k}}}{2T_{\mathrm{c}}}\right) - \frac{m}{k^2}\right] = 0$$

${}^{1}S_{0}$ critical temperature

HT, T. Hatsuda, P. van Wyk, and Y. Ohashi, Sci. Rep. **9**, 18477 (2019).



● Lattice QMC: T. Abe and R. Seki, Phys. Rev. C **79**, 054002 (2009).

$V_{\text{low }k}$: S. Ramanan and M. Urban, Phys. Rev. C **88**, 054315 (2013)

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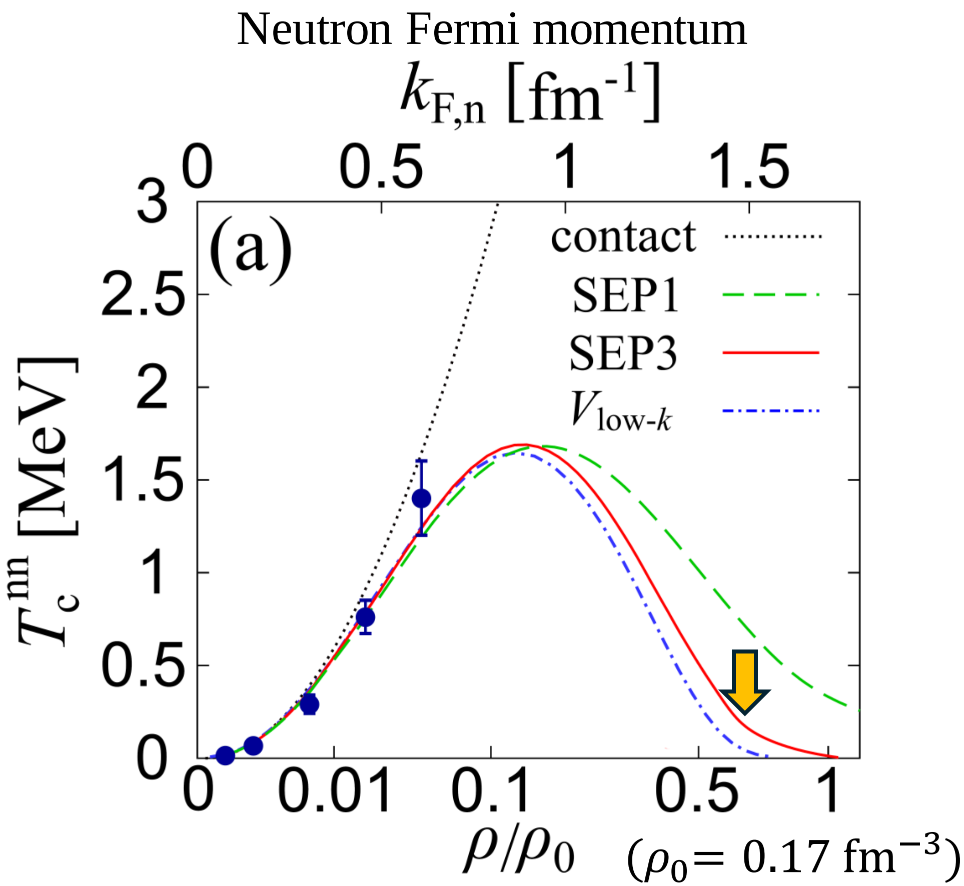
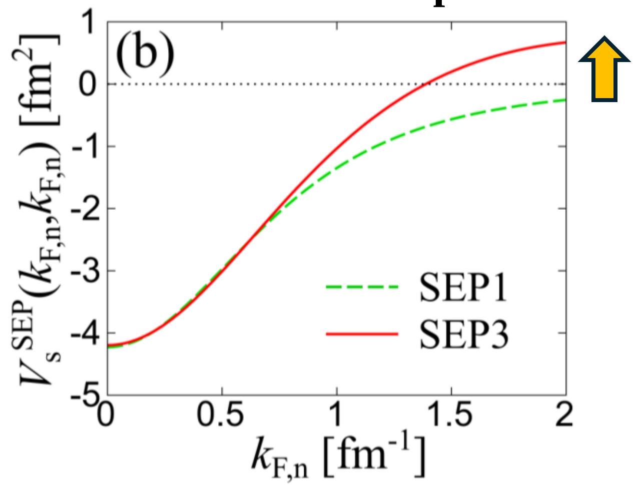


Multi-rank with repulsion



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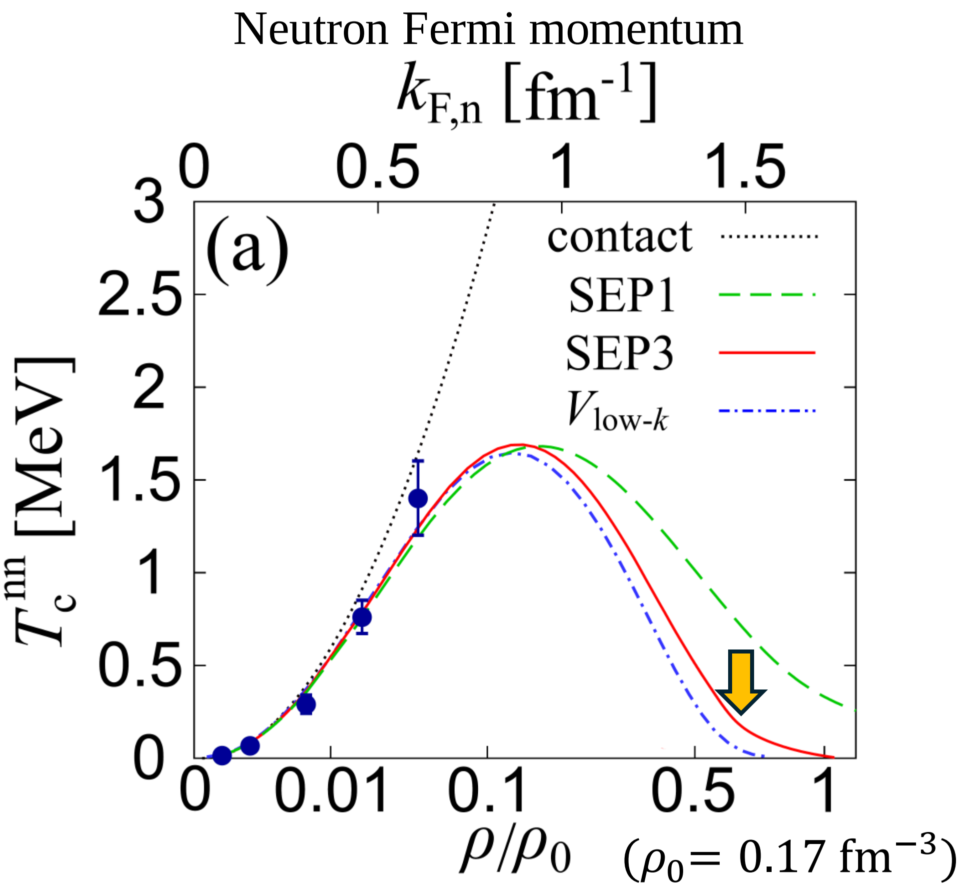
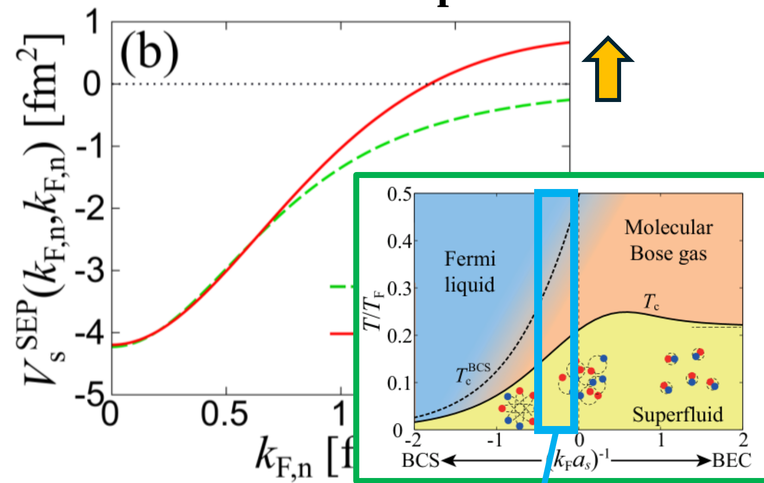
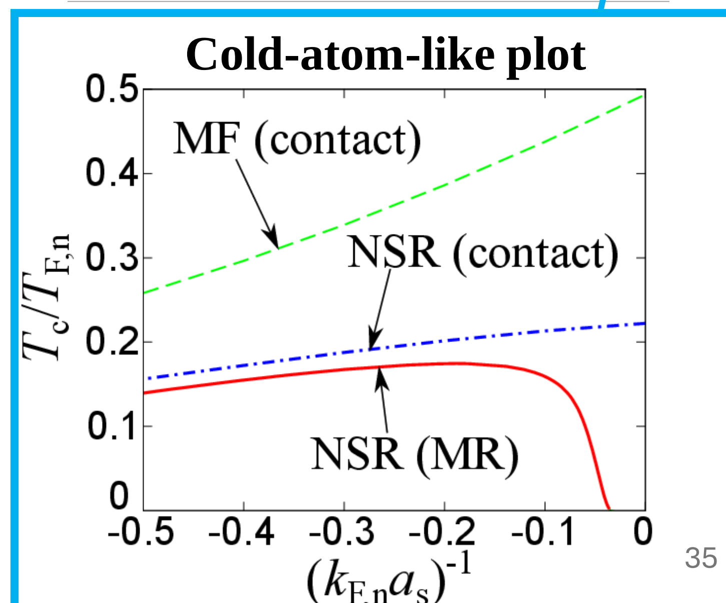


Multi-rank with repulsion





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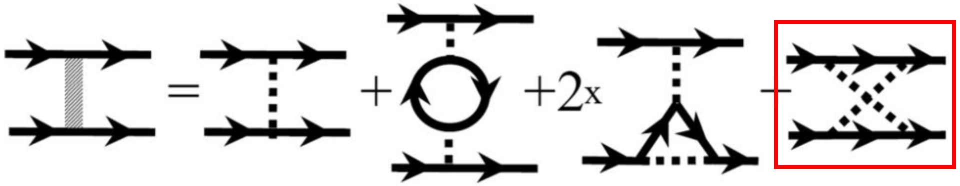
$V_{\text{low }k}$: S. Ramanan and M. Urban, Phys. Rev. C **88**, 054315 (2013).

Another effect beyond BCS: Gor'kov-Melik-Barkhudanov (GMB)

L. P. Gorkov and T. M. Melik-Barkhudarov, Sov. Phys. JETP **13**, 1018 (1961).

Screened attractive interaction by particle-hole fluctuations



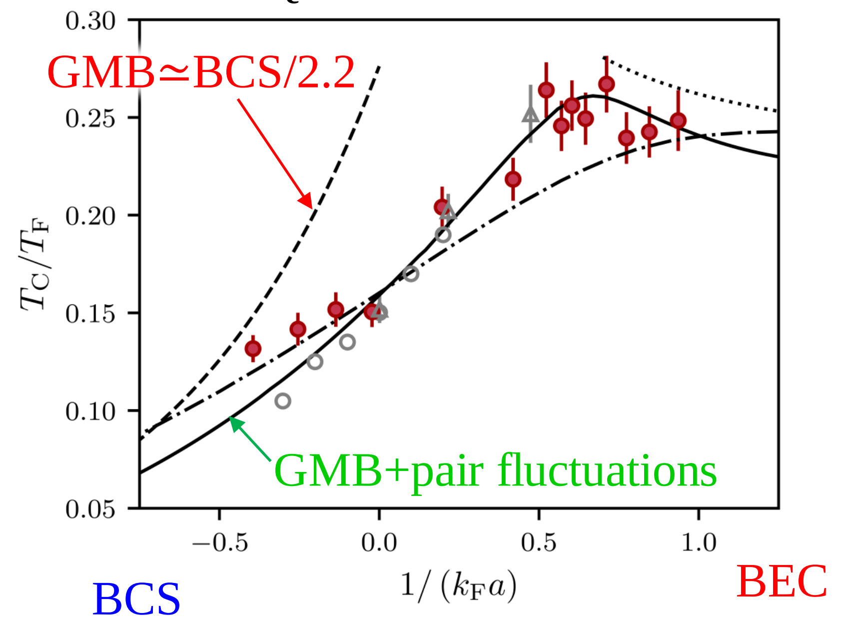
A. Chubukov, et al., PRB **93**, 174516 (2016)

T_c is suppressed by a factor ~**2.2**

$$T_c^{(\mathrm{GMB})} \approx \left(\frac{2}{e}\right)^{7/3} \frac{\gamma}{\pi} T_F e^{\pi/2k_F a_s} \approx 0.28 T_F e^{\pi/2k_F a_s}$$

$$\simeq \frac{1}{(4e)^{1/3}} T_c^{(\mathrm{BCS})} \simeq \frac{1}{2.2} T_c^{(\mathrm{BCS})}$$

Latest experimental and theoretical results of T_c in the BCS-BEC crossover



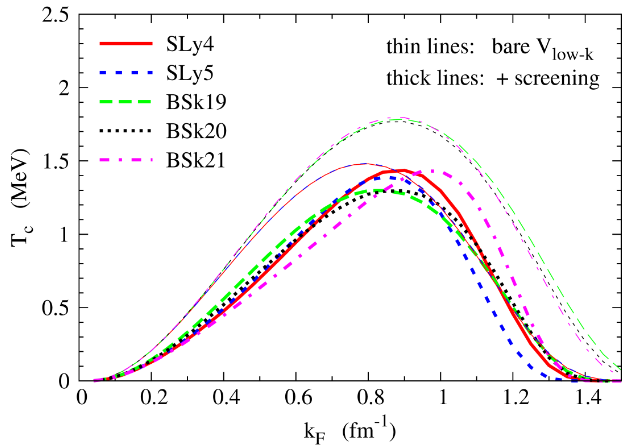
Interaction strength

M. Link, et al., PRL **130**, 203401 (2023).

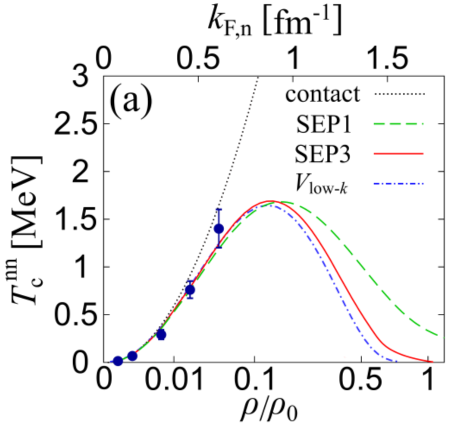
GMB+pair fluctuations : L. Pisani, A, Perali, P. Pieri, and G. C. Strinati, PRB **97**, 014528 (2018).

Another effect beyond BCS: Gor'kov-Melik-Barkhudanov (GMB)

Screening effect in $^{1}S_0$ neutron superfluid



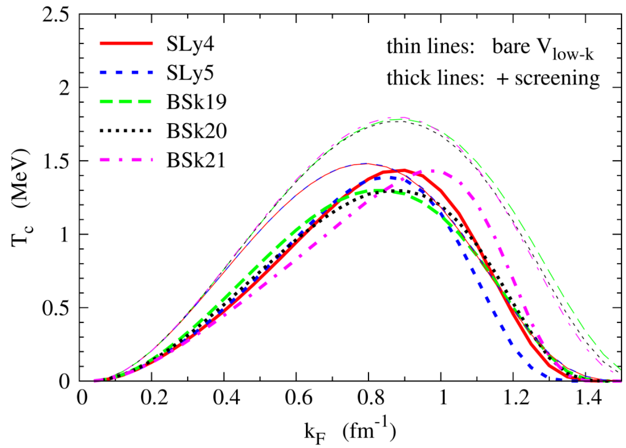
M. Urban *et al.*, PRC **101**, 035803 (2020).



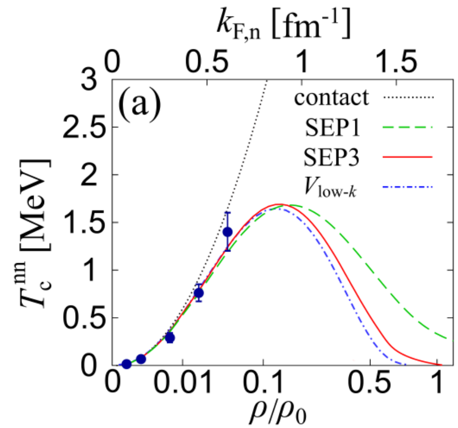
Understimating T_c compared to QMC?

Another effect beyond BCS: Gor'kov-Melik-Barkhudanov (GMB)

Screening effect in $^1S_0$ neutron superfluid



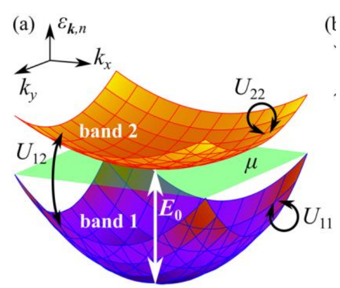
M. Urban *et al.*, PRC **101**, 035803 (2020).

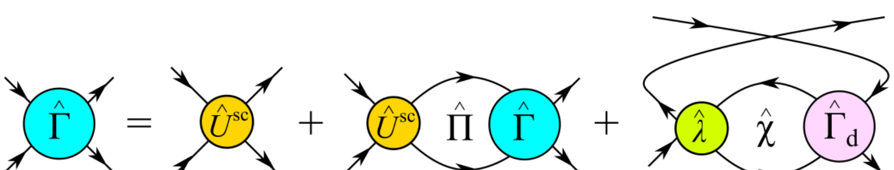


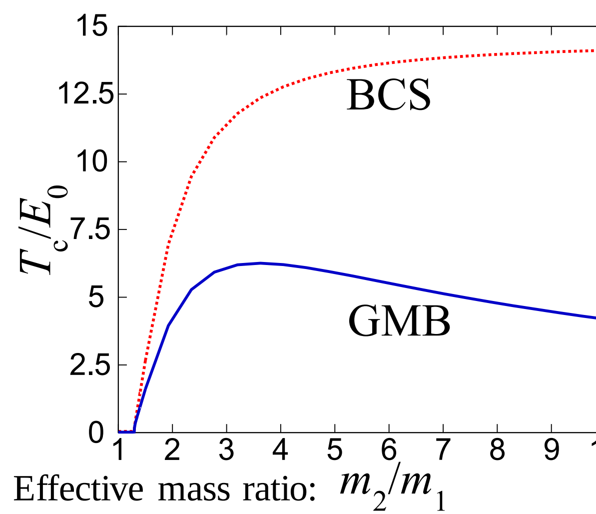
Understimating T_c compared to QMC?

Enhanced GMB screening near the flat band

[HT](), et al., Phys. Rev. B **109**, L140504 (2024)

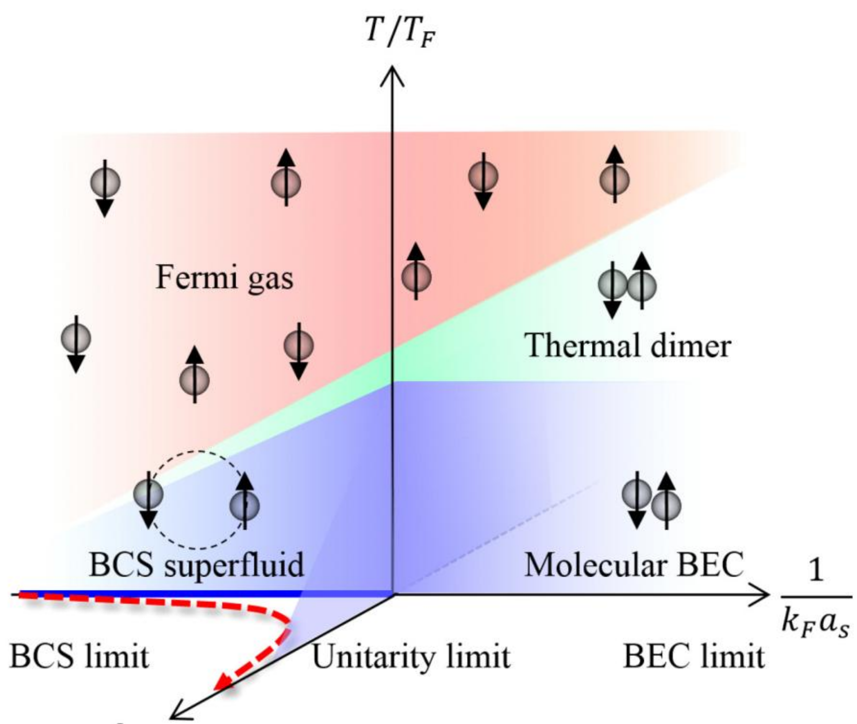


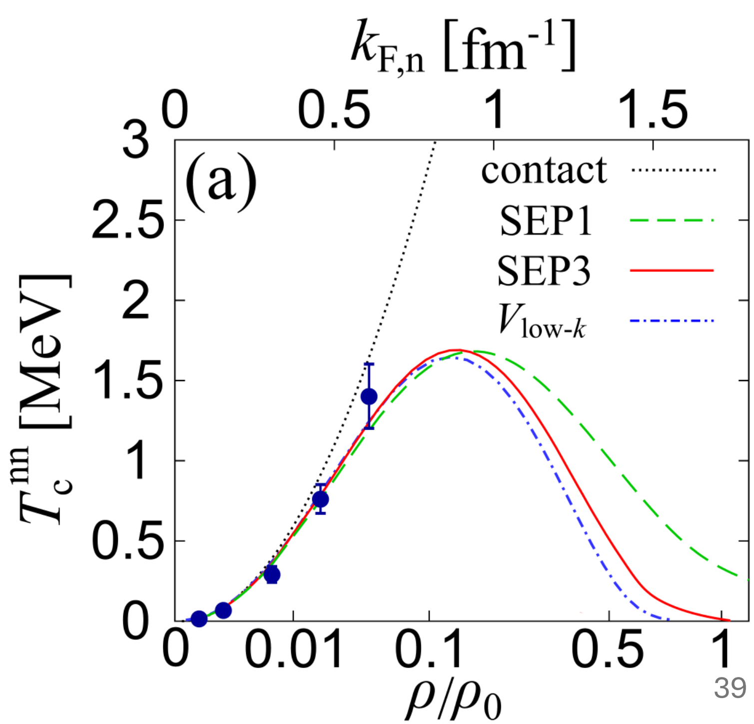




Short summary of neutron superfluid

- The difference between an ultracold Fermi gas and neutron matter, that is, the non-local interaction has been addressed.
- While it is still challenging to control r_{eff} in ultracold atom experiments, one can bridge two systems by using a reliable theory.
- The finite-range extension of the BCS-Eagles Leggett and NSR theories has been presented.





Outline

- Difference between a ultracold Fermi gas and neutron matter
- Neutron superfluid
- Neutron-proton pairing and proton superconductivity
- Short summary

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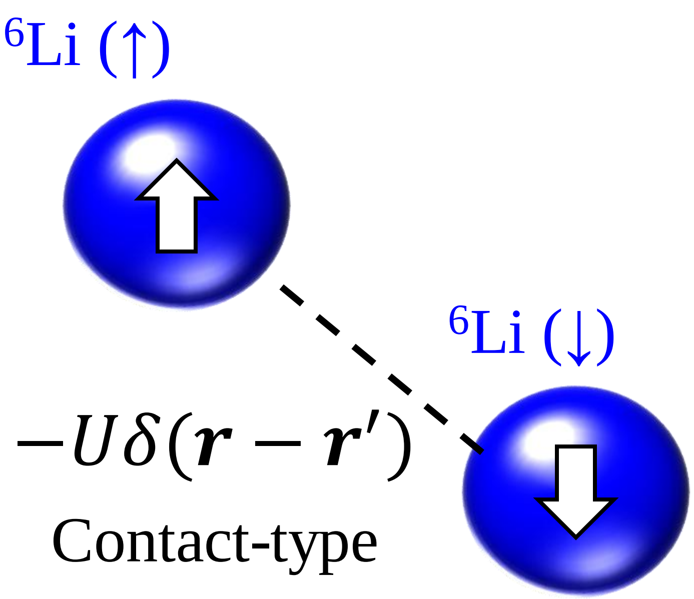
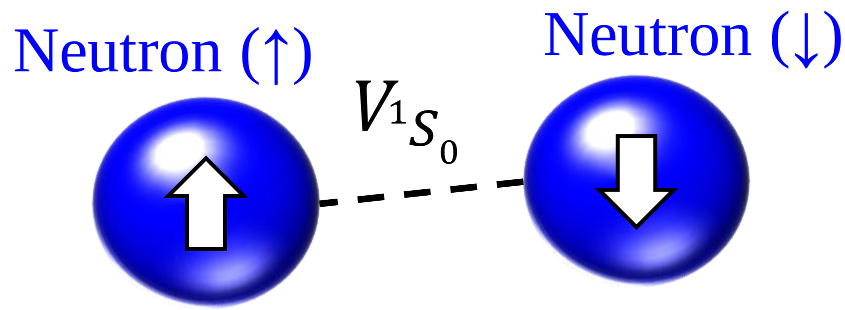
Difference between cold atoms and "neutron star matter"

Ultracold Fermi atomic gas		**Pure neutron matter**
Zero-range contact potential 2-component Fermi superfluid	⇔	Nuclear force (non-local) Neutrons (2-component)

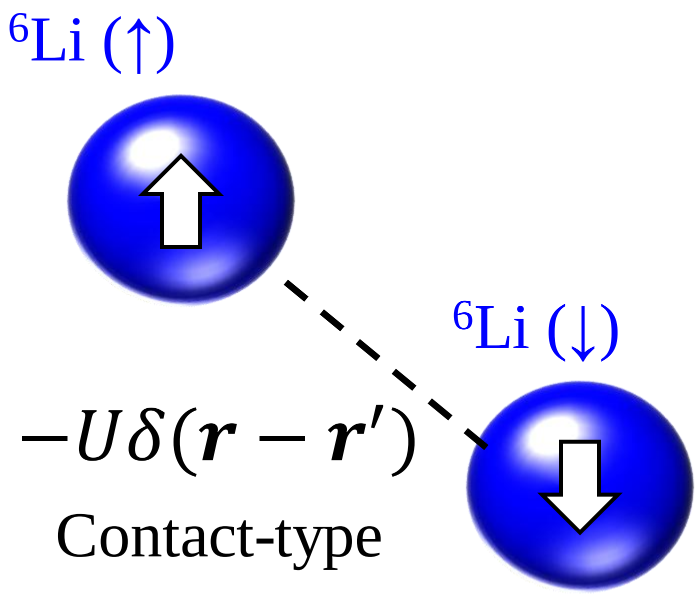


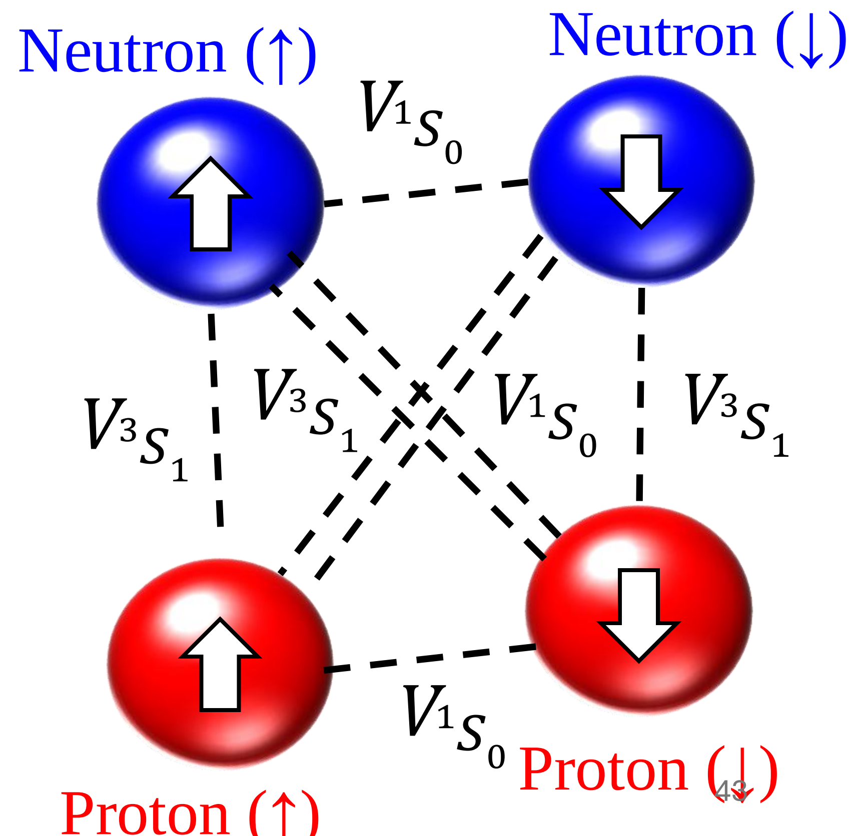


Difference between cold atoms and “neutron star matter”

Ultracold Fermi atomic gas | **Asymmetric nuclear matter**

Ultracold Fermi atomic gas	⇔	Asymmetric nuclear matter
Zero-range contact potential 2-component Fermi superfluid	⇔	Nuclear force (non-local) Neutron+Proton (4-component)

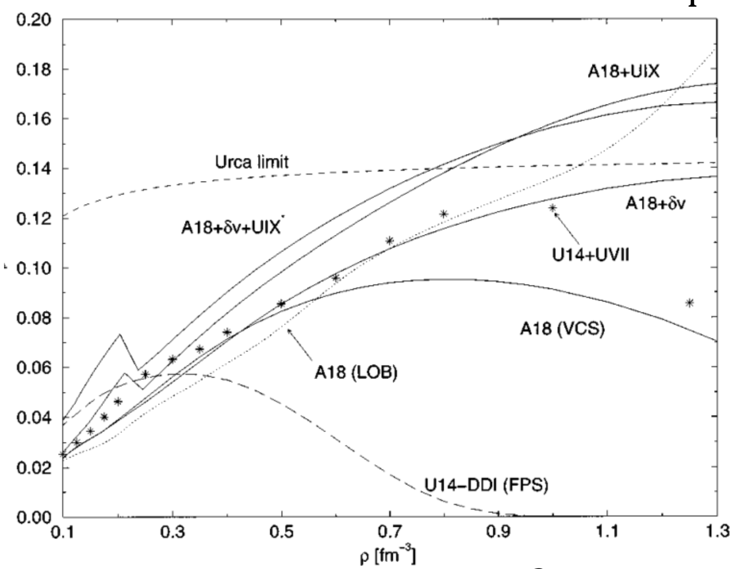




Finite proton fraction in neutron star matter

Proton fraction $\mathrm{Y_p} = \dfrac{\rho_\mathrm{p}}{\rho_\mathrm{n} + \rho_\mathrm{p}}$

$\mathrm{Y_p}$



Density ρ [fm^{-3}]

APR, PRC, **58**, 1804 (1998)

Typically $\mathrm{Y_p} \simeq 0 \sim 0.1$

$\boldsymbol{\beta}$ equilibrium

$$\mu_\mathrm{n} = \mu_\mathrm{p} + \mu_\mathrm{e}$$

Charge neutrality

$$\rho_\mathrm{p} = \rho_\mathrm{e}$$

Fermi degenerate chemical potential

$$\mu_\mathrm{n} = \frac{\left(3\pi^2\rho_\mathrm{n}\right)^{2/3}}{2m} \quad \mu_\mathrm{p} = \frac{\left(3\pi^2\rho_\mathrm{p}\right)^{2/3}}{2m}$$

$$\mu_\mathrm{e} = \left(3\pi^2\rho_\mathrm{e}\right)^{1/3} \text{ (relativistic, } m_\mathrm{e} \simeq 0\text{)}$$

Equation for $\mathrm{Y_p}$ in the non-interacting case

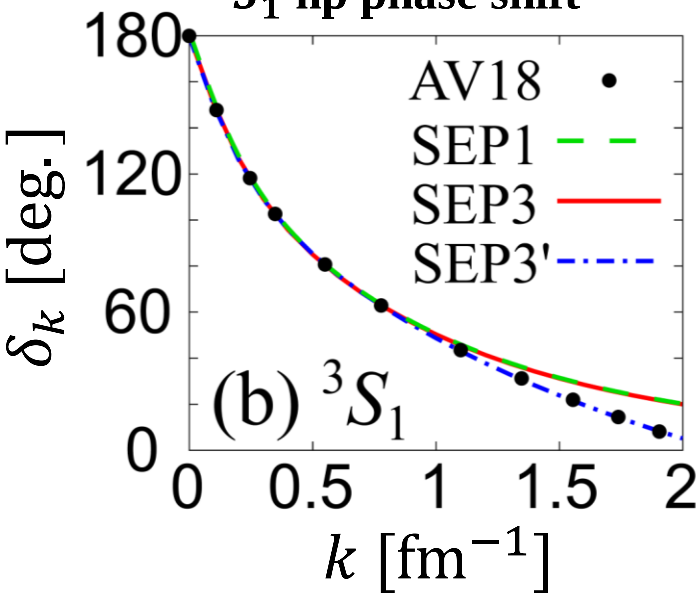
$$\left(1 - \mathrm{Y_p}\right)^{2/3} = \mathrm{Y_p^{2/3}} + \frac{2m\mathrm{Y_p^{1/3}}}{(3\pi^2\rho)^{1/3}}$$

$* \rho = \rho_\mathrm{n} + \rho_\mathrm{p}$

$\hbar c = 1 \quad m \simeq 939$ MeV

Finite proton fraction in neutron star matter

$^3S_1$ np phase shift



$r_{\mathrm{eff}} = 1.76$ fm

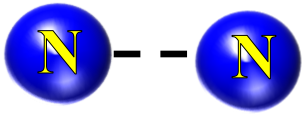
$a_s = 5.42$ fm

Deuteron binding energy

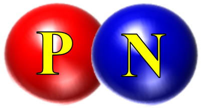
$E_{\mathrm{b}} = 2.22$ MeV

Dineutron correlation

Deuteron formation



・Cooper pairing without the density imbalance
・No binding energy



・Large density imbalance due to small proton fraction
・Finite binding energy

How strong spin-triplet np pairing affects $^1S_0$ superfluidity?

Neutron-proton mixture

Hamiltonian

$$H = \sum_{\tau_z} \sum_{s_z} \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}\tau_z} c^{\dagger}_{\boldsymbol{k}s_z\tau_z} c_{\boldsymbol{k}s_z\tau_z}$$

spin-singlet isospin-triplet (isovector) interaction

$$+\frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{k'},\boldsymbol{P}} \sum_{T_z=-1}^{+1} C^{\dagger}_{T_z}(\boldsymbol{k},\boldsymbol{P}) V_{\mathrm{s}}(\boldsymbol{k},\boldsymbol{k'}) C_{T_z}(\boldsymbol{k},\boldsymbol{P})$$

spin-triplet isospin-singlet (isoscalar) interaction

$$+\frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{k'},\boldsymbol{P}} \sum_{S_z=-1}^{+1} D^{\dagger}_{S_z}(\boldsymbol{k},\boldsymbol{P}) V_{\mathrm{t}}(\boldsymbol{k},\boldsymbol{k'}) D_{S_z}(\boldsymbol{k},\boldsymbol{P})$$

C_{T_z}, D_{S_z}: pair operators in their channels

- Kinetic energy: $\xi_{\boldsymbol{k}\tau_z} = \frac{k^2}{2m} - \mu_{\tau_z}$
- Nucleon spin and isospin : s, τ
- Nucleon mass: $m = 939\mathrm{MeV}$
- Chemical potential: μ_{τ_z}
- Annihilation operator: $c_{\boldsymbol{k}s_z\tau_z}$
- Interaction potential: $V_{\mathrm{S,T}}(\boldsymbol{k},\boldsymbol{k'})$

Neutron-proton mixture

Hamiltonian

$$H = \sum_{\tau_z} \sum_{s_z} \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}\tau_z} c^{\dagger}_{\boldsymbol{k}s_z\tau_z} c_{\boldsymbol{k}s_z\tau_z}$$

spin-singlet isospin-triplet (isovector) interaction

$$+\frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{k'},\boldsymbol{P}} \sum_{T_z=-1}^{+1} C^{\dagger}_{T_z}(\boldsymbol{k},\boldsymbol{P}) V_{\mathrm{s}}(\boldsymbol{k},\boldsymbol{k'}) C_{T_z}(\boldsymbol{k},\boldsymbol{P})$$

spin-triplet isospin-singlet (isoscalar) interaction

$$+\frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{k'},\boldsymbol{P}} \sum_{S_z=-1}^{+1} D^{\dagger}_{S_z}(\boldsymbol{k},\boldsymbol{P}) V_{\mathrm{t}}(\boldsymbol{k},\boldsymbol{k'}) D_{S_z}(\boldsymbol{k},\boldsymbol{P})$$

$$C_{T_z}(\boldsymbol{k},\boldsymbol{P}) = \sum_{s_z,s'_z,\tau_z,\tau'_z} \left\langle \frac{1}{2},\frac{1}{2};s_z,s'_z \middle| 0,0 \right\rangle \left\langle \frac{1}{2},\frac{1}{2};\tau_z,\tau'_z \middle| 1,T_z \right\rangle c_{-\boldsymbol{k}+\frac{\boldsymbol{P}}{2}\,s_z\tau_z} c_{\boldsymbol{k}+\frac{\boldsymbol{P}}{2}\,s'_z\tau'_z}$$

$$D_{S_z}(\boldsymbol{k},\boldsymbol{P}) = \sum_{s_z,s'_z,\tau_z,\tau'_z} \left\langle \frac{1}{2},\frac{1}{2};s_z,s'_z \middle| 1,S_z \right\rangle \left\langle \frac{1}{2},\frac{1}{2};\tau_z,\tau'_z \middle| 0,0 \right\rangle c_{-\boldsymbol{k}+\frac{\boldsymbol{P}}{2}\,s_z\tau_z} c_{\boldsymbol{k}+\frac{\boldsymbol{P}}{2}\,s'_z\tau'_z}$$

Neutron-proton mixture

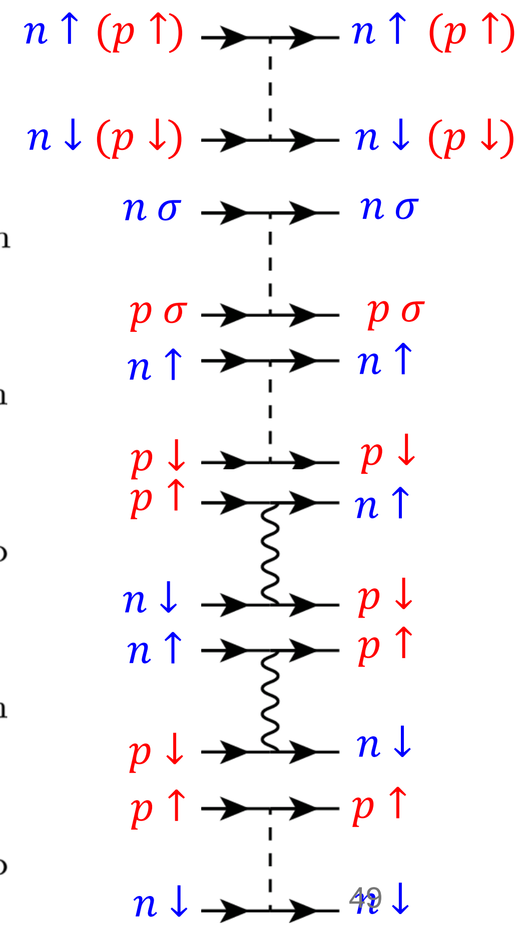
$s_z = +(-)1/2 \coloneqq \uparrow (\downarrow) \qquad \tau_z = +(-)1/2 \coloneqq \mathrm{n}(\mathrm{p})$

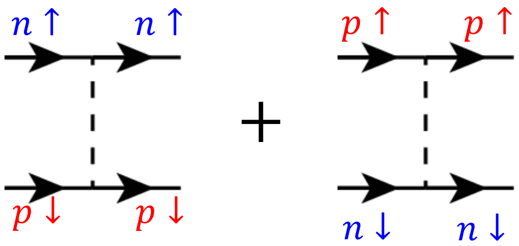
$$
\begin{aligned}
H &= \sum_{\sigma=\uparrow,\downarrow} \sum_{\zeta=\mathrm{n},\mathrm{p}} \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k},\zeta} c^\dagger_{\boldsymbol{k},\sigma,\zeta} c_{\boldsymbol{k},\sigma,\zeta} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \sum_{\zeta=\mathrm{n},\mathrm{p}} V_{\mathrm{s}} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\zeta} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\zeta} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\zeta} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\zeta} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \sum_{\sigma=\uparrow,\downarrow} V_{\mathrm{t}} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\sigma,\mathrm{n}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\sigma,\mathrm{p}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\sigma,\mathrm{p}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\sigma,\mathrm{n}} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \frac{V_{\mathrm{s}}+V_{\mathrm{t}}}{2} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\mathrm{n}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\mathrm{p}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\mathrm{p}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\mathrm{n}} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \frac{V_{\mathrm{s}}-V_{\mathrm{t}}}{2} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\mathrm{n}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\mathrm{p}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\mathrm{n}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\mathrm{p}} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \frac{V_{\mathrm{s}}-V_{\mathrm{t}}}{2} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\mathrm{p}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\mathrm{n}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\mathrm{p}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\mathrm{n}} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \frac{V_{\mathrm{s}}+V_{\mathrm{t}}}{2} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\mathrm{p}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\mathrm{n}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\mathrm{n}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\mathrm{p}}
\end{aligned}
$$

Neutron-proton mixture

$$s_z = +(-)1/2 \coloneqq \uparrow (\downarrow) \qquad \tau_z = +(-)1/2 \coloneqq \mathrm{n}(\mathrm{p})$$

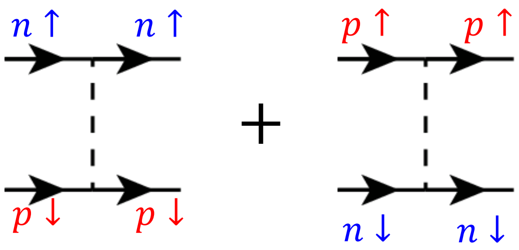
$$\begin{aligned}
H =& \sum_{\sigma=\uparrow,\downarrow} \sum_{\zeta=\mathrm{n,p}} \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k},\zeta} c^\dagger_{\boldsymbol{k},\sigma,\zeta} c_{\boldsymbol{k},\sigma,\zeta} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \sum_{\zeta=\mathrm{n,p}} V_{\mathrm{s}} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\zeta} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\zeta} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\zeta} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\zeta} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \sum_{\sigma=\uparrow,\downarrow} V_{\mathrm{t}} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\sigma,\mathrm{n}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\sigma,\mathrm{p}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\sigma,\mathrm{p}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\sigma,\mathrm{n}} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \frac{V_{\mathrm{s}}+V_{\mathrm{t}}}{2} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\mathrm{n}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\mathrm{p}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\mathrm{p}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\mathrm{n}} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \frac{V_{\mathrm{s}}-V_{\mathrm{t}}}{2} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\mathrm{n}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\mathrm{p}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\mathrm{n}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\mathrm{p}} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \frac{V_{\mathrm{s}}-V_{\mathrm{t}}}{2} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\mathrm{p}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\mathrm{n}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\mathrm{p}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\mathrm{n}} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \frac{V_{\mathrm{s}}+V_{\mathrm{t}}}{2} c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow,\mathrm{p}} c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow,\mathrm{n}} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow,\mathrm{n}} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow,\mathrm{p}}
\end{aligned}$$





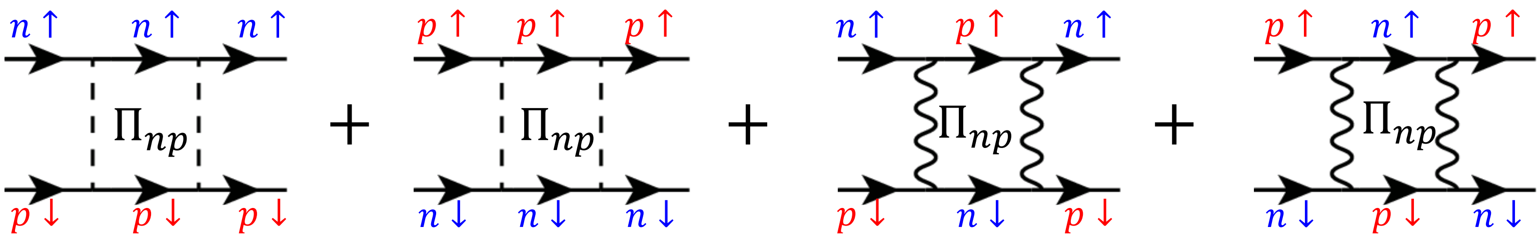
1st

$$\frac{V_{\mathrm{s}} + V_{\mathrm{t}}}{2} + \frac{V_{\mathrm{s}} + V_{\mathrm{t}}}{2} = V_{\mathrm{s}} + V_{\mathrm{t}}$$



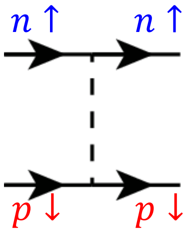
1st

$$\frac{V_\mathrm{s}+V_\mathrm{t}}{2}+\frac{V_\mathrm{s}+V_\mathrm{t}}{2}=V_\mathrm{s}+V_\mathrm{t}$$



2nd

$$2\left(\frac{V_\mathrm{s}+V_\mathrm{t}}{2}\right)^2\Pi_{np}+2\left(\frac{V_\mathrm{s}-V_\mathrm{t}}{2}\right)^2\Pi_{np}=\left(V_\mathrm{s}{}^2+V_\mathrm{t}{}^2\right)\Pi_{np}$$



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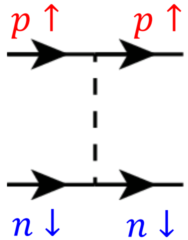
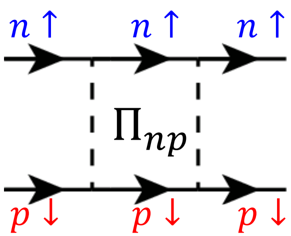


1st

$$\frac{V_\mathrm{s}+V_\mathrm{t}}{2}+\frac{V_\mathrm{s}+V_\mathrm{t}}{2}=V_\mathrm{s}+V_\mathrm{t}$$



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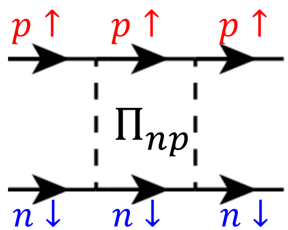

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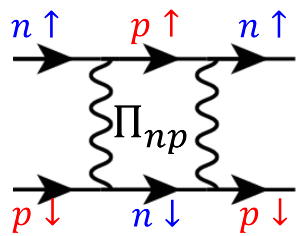

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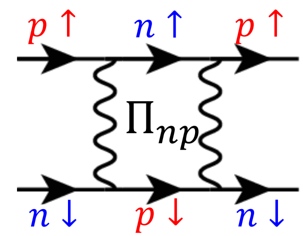
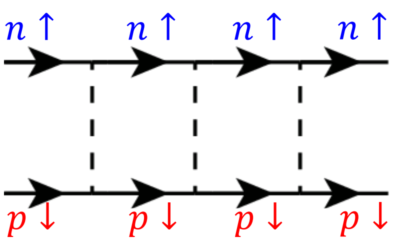
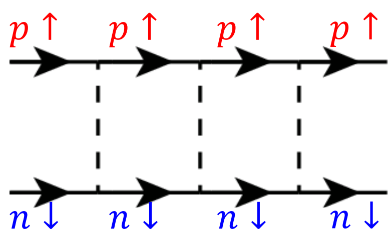


2nd

$$2\left(\frac{V_\mathrm{s}+V_\mathrm{t}}{2}\right)^2\Pi_{np}+2\left(\frac{V_\mathrm{s}-V_\mathrm{t}}{2}\right)^2\Pi_{np}=\left(V_\mathrm{s}{}^2+V_\mathrm{t}{}^2\right)\Pi_{np}$$



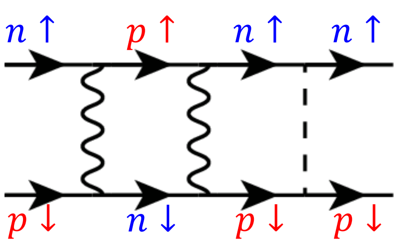
+



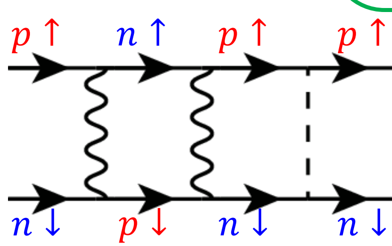
3rd

$$2\left(\frac{V_\mathrm{s}+V_\mathrm{t}}{2}\right)^3\Pi_{np}+6\left(\frac{V_\mathrm{s}+V_\mathrm{t}}{2}\right)\left(\frac{V_\mathrm{s}+V_\mathrm{t}}{2}\right)^2\Pi_{np}$$
$$=\left(V_\mathrm{s}{}^3+V_\mathrm{t}{}^3\right)\Pi_{np}^2$$

+



$\times 3$ +



$\times 3$

***np T* matrix**

$$\boxed{T_{np}=\frac{V_\mathrm{s}}{1-V_\mathrm{s}\Pi_{np}}+\frac{V_\mathrm{t}}{1-V_t\Pi_{np}}}$$

NSR theory in asymmetric nuclear matter

Neutron number density

$$\rho_{\mathrm{NSR,n}} = \sum_{\boldsymbol{k},,s_z} f(\xi^*_{\boldsymbol{k}\mathrm{n}}) - \frac{1}{\beta}\sum_{\boldsymbol{P},\ell}[T_{\mathrm{nn}}(\boldsymbol{P},i\nu_\ell) - g_{\mathrm{nn}}]\frac{\partial \Pi_{\mathrm{nn}}(\boldsymbol{P},i\nu_\ell)}{\partial \mu_{\mathrm{n}}} e^{i\nu_\ell\delta}$$

Isovector nn

$$- \frac{1}{\beta}\sum_{\boldsymbol{P},\ell}[T^{\mathrm{t}}_{\mathrm{np}}(\boldsymbol{P},i\nu_\ell) - g^s_{\mathrm{np}}]\frac{\partial \Pi^s_{\mathrm{np}}(\boldsymbol{P},i\nu_\ell)}{\partial \mu_{\mathrm{n}}} e^{i\nu_\ell\delta}$$

Isovector np

$$- \frac{3}{\beta}\sum_{\boldsymbol{P},\ell}[T^{\mathrm{t}}_{\mathrm{np}}(\boldsymbol{P},i\nu_\ell) - g^s_{\mathrm{np}}]\frac{\partial \Pi^t_{\mathrm{np}}(\boldsymbol{P},i\nu_\ell)}{\partial \mu_{\mathrm{n}}} e^{i\nu_\ell\delta}$$

Isoscalar np

Proton number density

$$\rho_{\mathrm{NSR,p}} = \sum_{\boldsymbol{k},,s_z} f(\xi^*_{\boldsymbol{k}\mathrm{p}}) - \frac{1}{\beta}\sum_{\boldsymbol{P},\ell}[T_{\mathrm{pp}}(\boldsymbol{P},i\nu_\ell) - g_{\mathrm{pp}}]\frac{\partial \Pi_{\mathrm{pp}}(\boldsymbol{P},i\nu_\ell)}{\partial \mu_{\mathrm{p}}} e^{i\nu_\ell\delta}$$

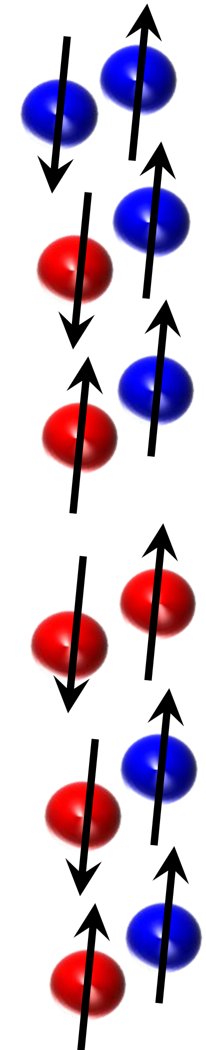
Isovector pp

$$- \frac{1}{\beta}\sum_{\boldsymbol{P},\ell}[T^{\mathrm{t}}_{\mathrm{np}}(\boldsymbol{P},i\nu_\ell) - g^s_{\mathrm{np}}]\frac{\partial \Pi^s_{\mathrm{np}}(\boldsymbol{P},i\nu_\ell)}{\partial \mu_{\mathrm{p}}} e^{i\nu_\ell\delta}$$

Isovector np

$$- \frac{3}{\beta}\sum_{\boldsymbol{P},\ell}[T^{\mathrm{t}}_{\mathrm{np}}(\boldsymbol{P},i\nu_\ell) - g^s_{\mathrm{np}}]\frac{\partial \Pi^t_{\mathrm{np}}(\boldsymbol{P},i\nu_\ell)}{\partial \mu_{\mathrm{p}}} e^{i\nu_\ell\delta}$$

Isoscalar np



NSR theory
in asymmetric nuclear matter

Nucleon number densities ➡ **Nucleon chemical potential**

$\rho_{\mathrm{NSR,n}} \quad \rho_{\mathrm{NSR,p}}$ $\qquad$ $\mu_{\mathrm{n}} \quad \mu_{\mathrm{p}}$

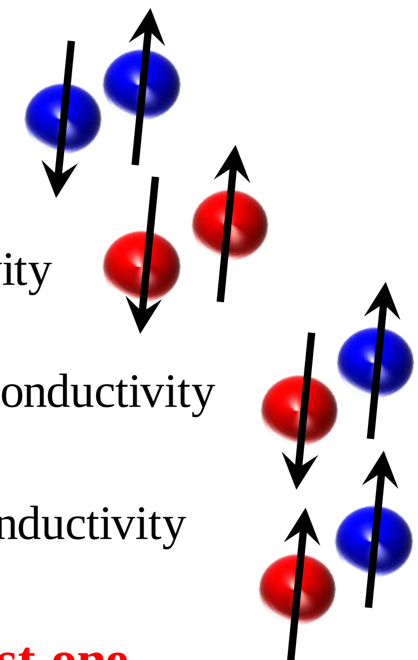
Thouless criterion of various pairing channels

$[T_{\mathrm{nn}}(\boldsymbol{P}=\mathbf{0}, i\nu_\ell = 0)]^{-1} = 0$:Isovector neutron superfluid

$\left[T_{\mathrm{pp}}(\boldsymbol{P}=\mathbf{0}, i\nu_\ell = 0)\right]^{-1} = 0$:Isovector proton superconductivity

$\left[T_{\mathrm{np}}^{s}(\boldsymbol{P}=\mathbf{0}, i\nu_\ell = 0)\right]^{-1} = 0$:Isovector neutron-proton superconductivity

$\left[T_{\mathrm{np}}^{t}(\boldsymbol{P}=\mathbf{0}, i\nu_\ell = 0)\right]^{-1} = 0$:Isoscalr neutron-proton superconductivity



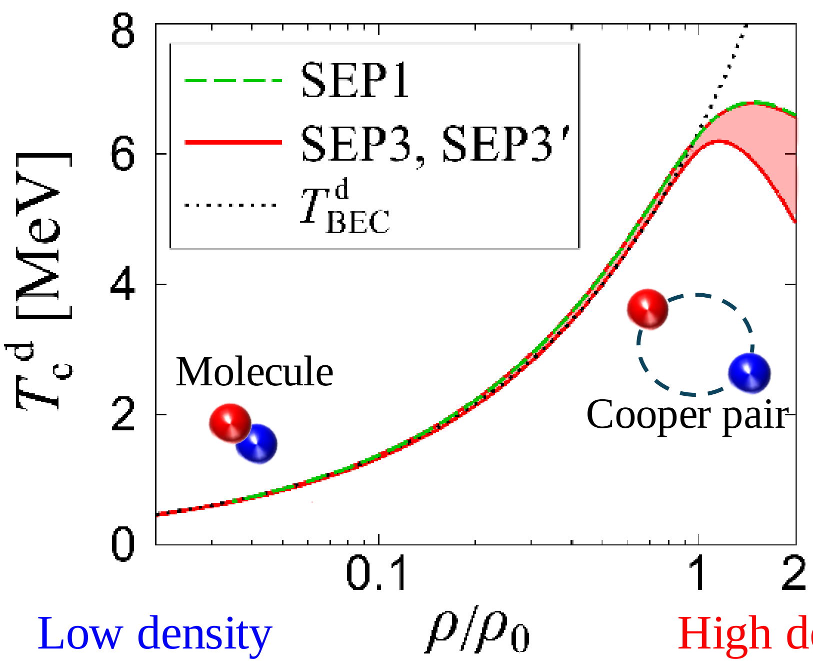
➡ **Look for the highest one**

Deuteron correlations in uniform matter

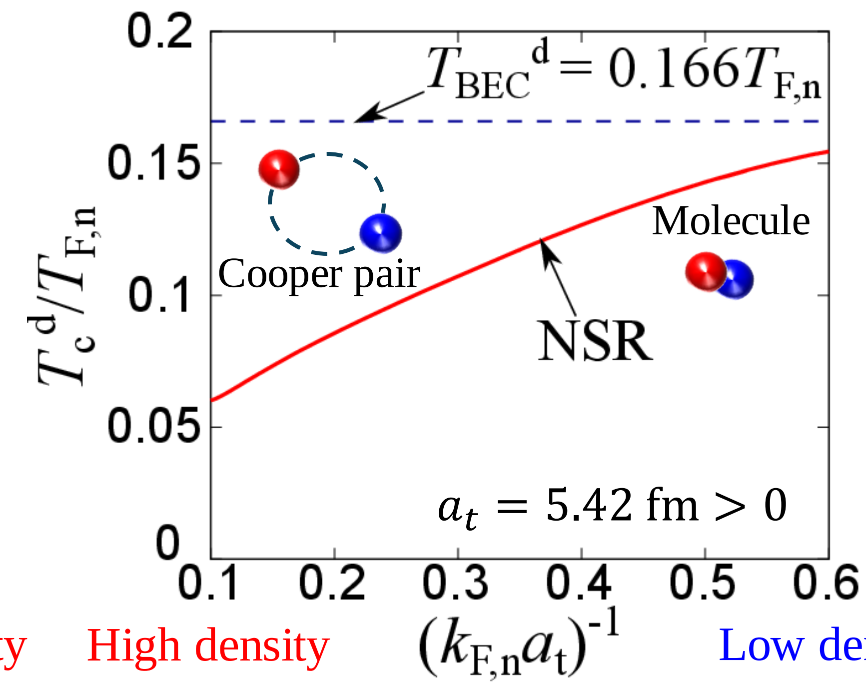
HT, T. Hatsuda, P. van Wyk, and Y. Ohashi, Sci. Rep. **9**, 18477 (2019).

- BEC-BCS crossover in symmetric nuclear matter

Deuteron condensation in SNM



Plot with cold-atom-like scale

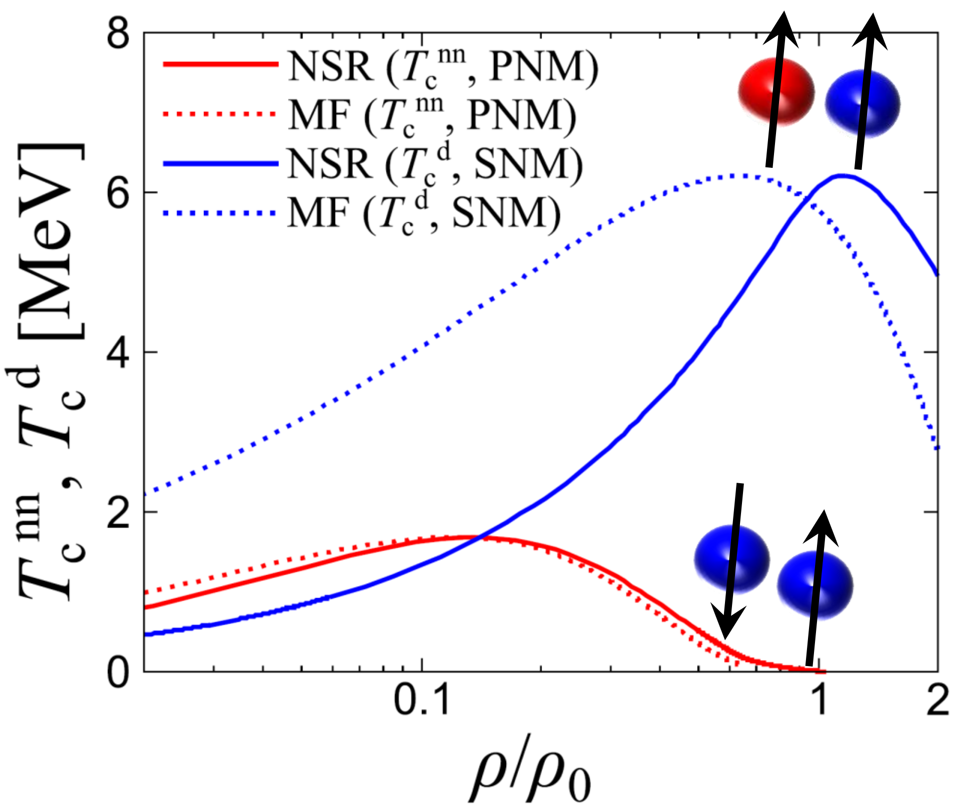


Low density High density High density Low density

Dilute deuteron BEC temperature: $\boxed{T_{\rm BEC}^{\rm d} = \frac{\pi}{m}\left[\frac{\rho_{\rm n}}{3\zeta(3/2)}\right]^{\frac{2}{3}}}$

Mean-field v.s. NSR

Fluctuation effects on T_c in SNM and PNM



Pairing fluctuation effects are more remarkable in np pairing than nn pairing

PNM (pure neutron matter) SNM (symmetric nuclear matter)

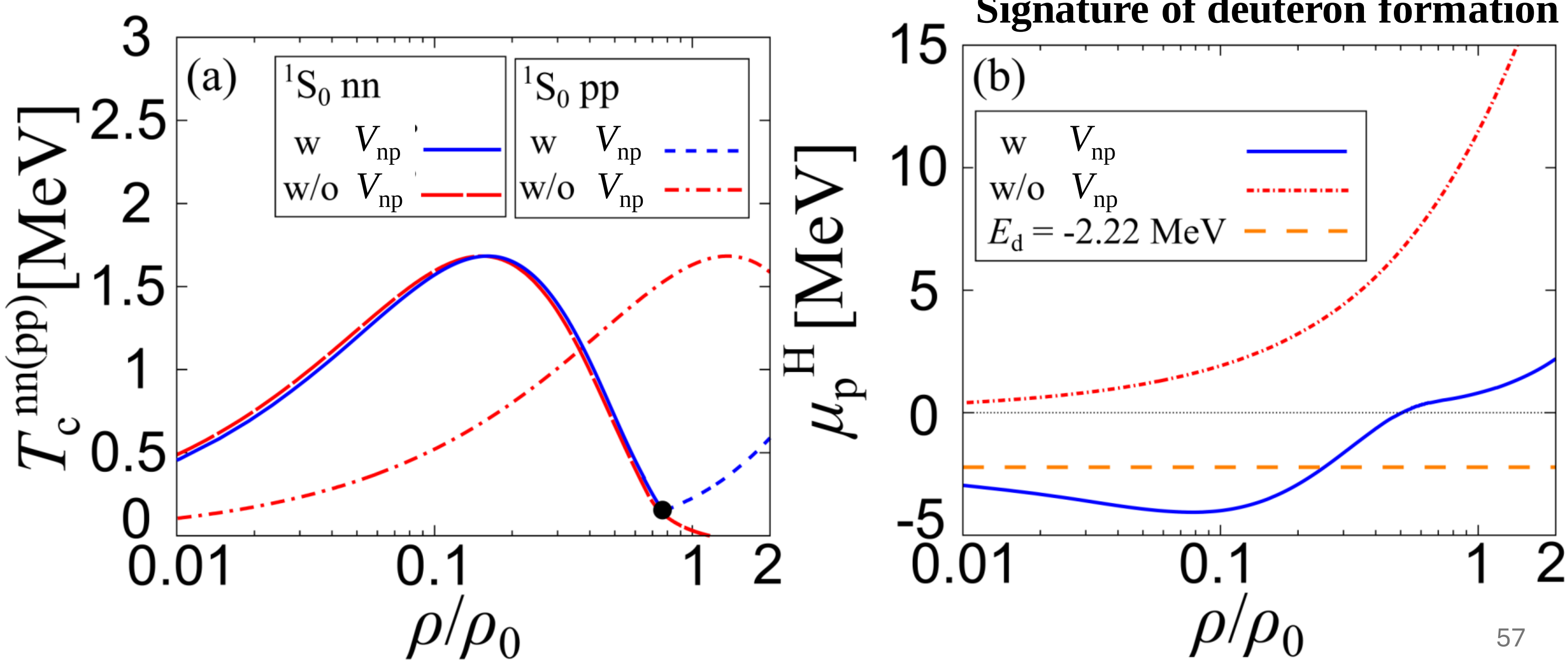
Interplay of pairing effects in asymmetric nuclear matter

Even without deuteron condensations, proton superconducting critical temperature is strongly affected by neutron-proton interaction.

Asymmetric nuclear matter: $Y_\mathrm{p} = 0.1$

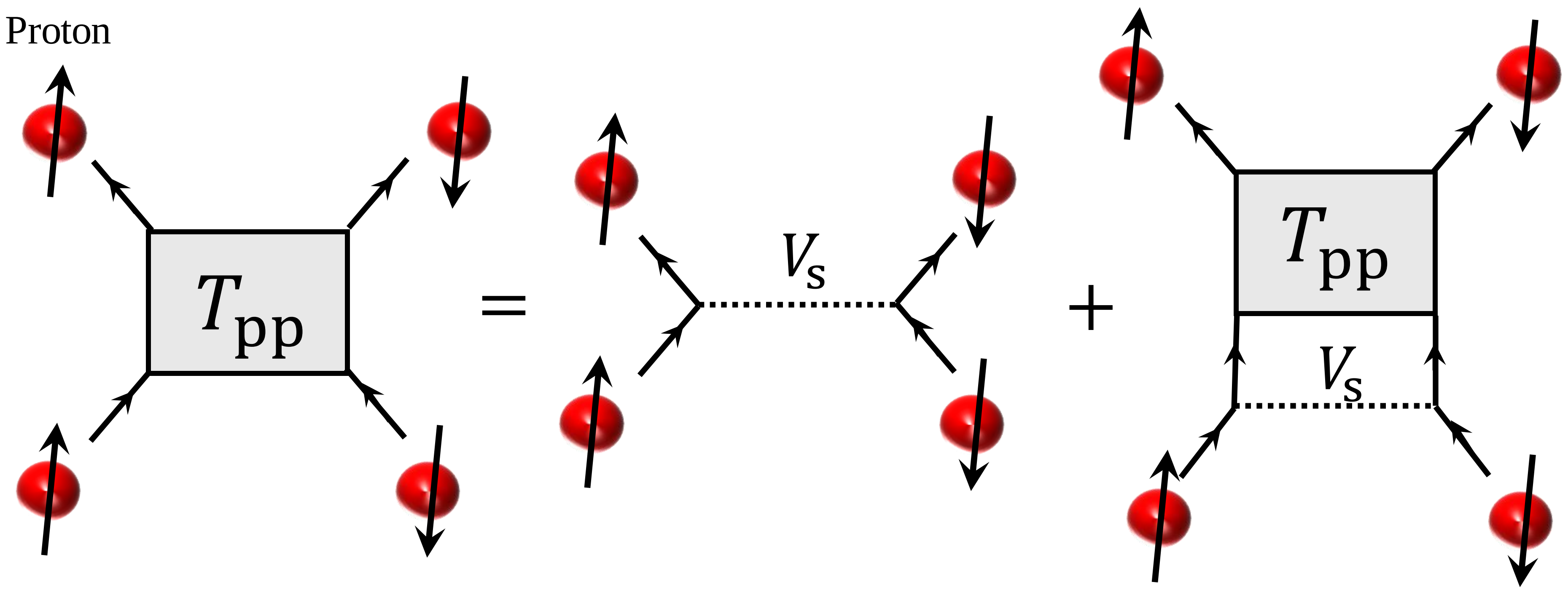
$\mu_\mathrm{p}^\mathrm{H} = \mu_\mathrm{p} - \Sigma_\mathrm{p}^\mathrm{H} \simeq -E_\mathrm{b}$

Signature of deuteron formation



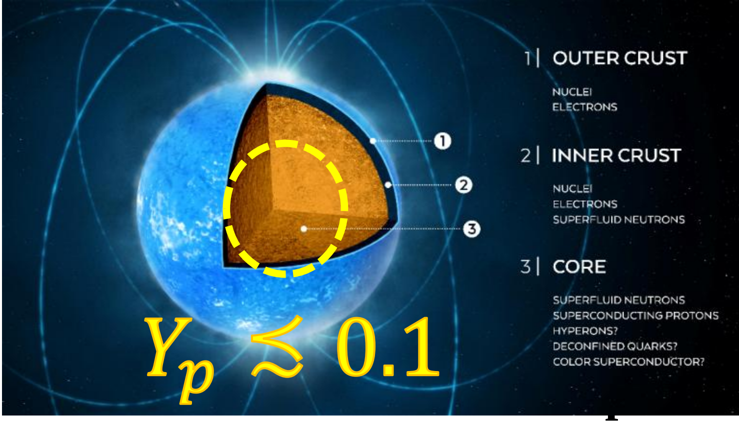
Proton pairing beyond NSR theory

Thouless criterion of proton superconductivity $\left[T_{\mathrm{pp}}(\boldsymbol{P}=\boldsymbol{0}, i\nu_{\ell}=0)\right]^{-1}=0$

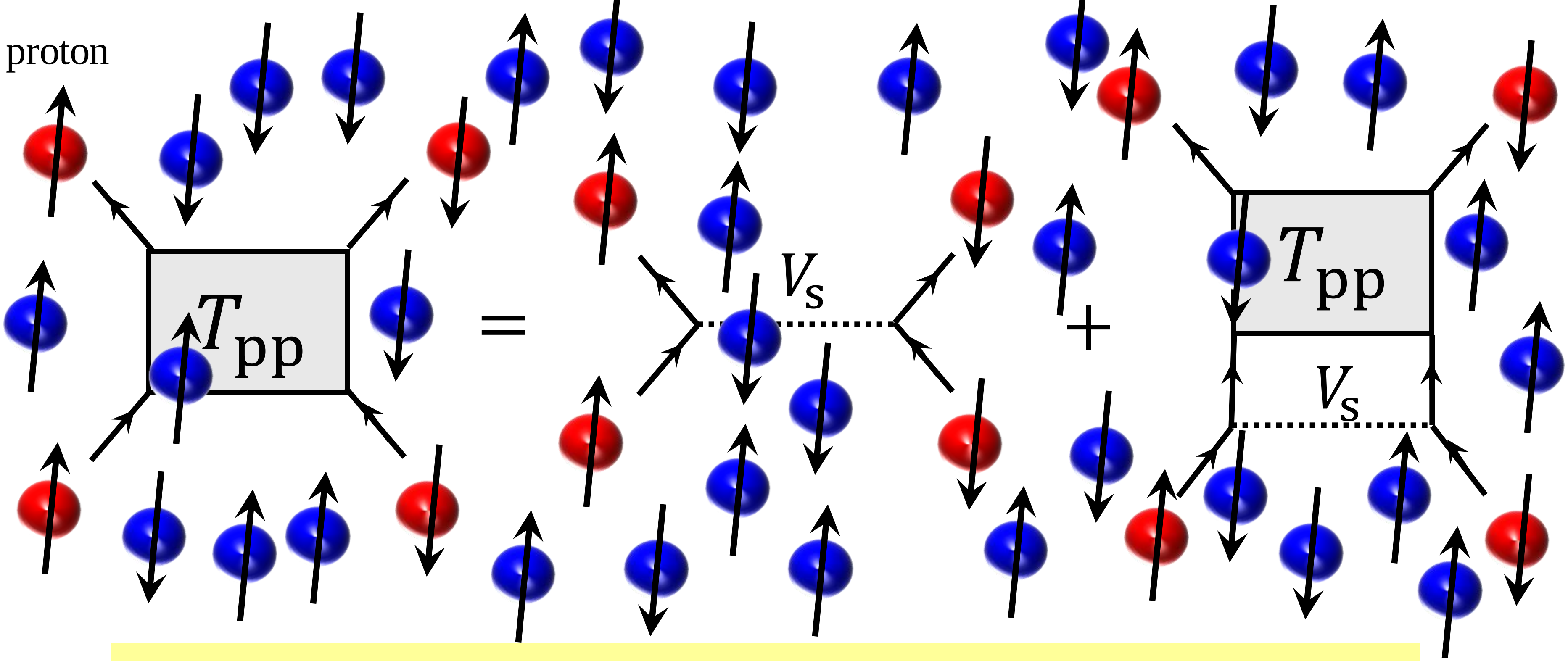


*In the NSR theory, the Thouless criterion of proton superconducting instability itself is insensitive to the existence of neutrons.

ng beyond NSR theory



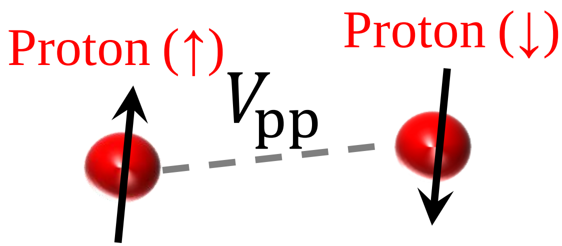
superconductivity $[T_{\mathrm{pp}}(\boldsymbol{P} = \mathbf{0}, i\nu_\ell = 0)]^{-1} = 0$

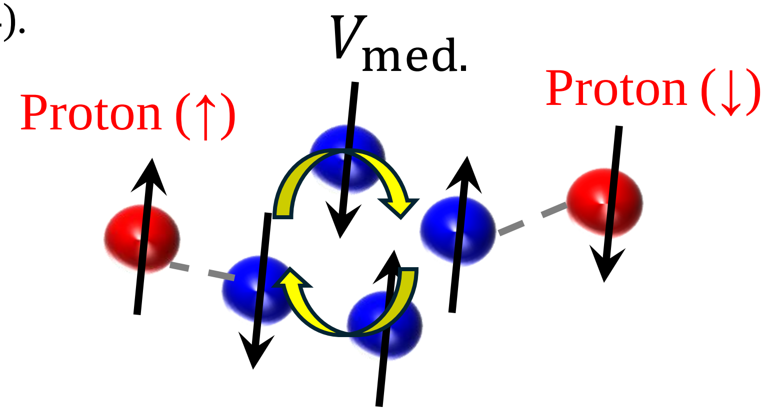


But we have more neutrons in neutron star matter!
➡How do many neutrons affect proton SC?

Neutron-mediated interaction between protons

HT, et al., Phys. Lett. B **851**, 138567 (2024).

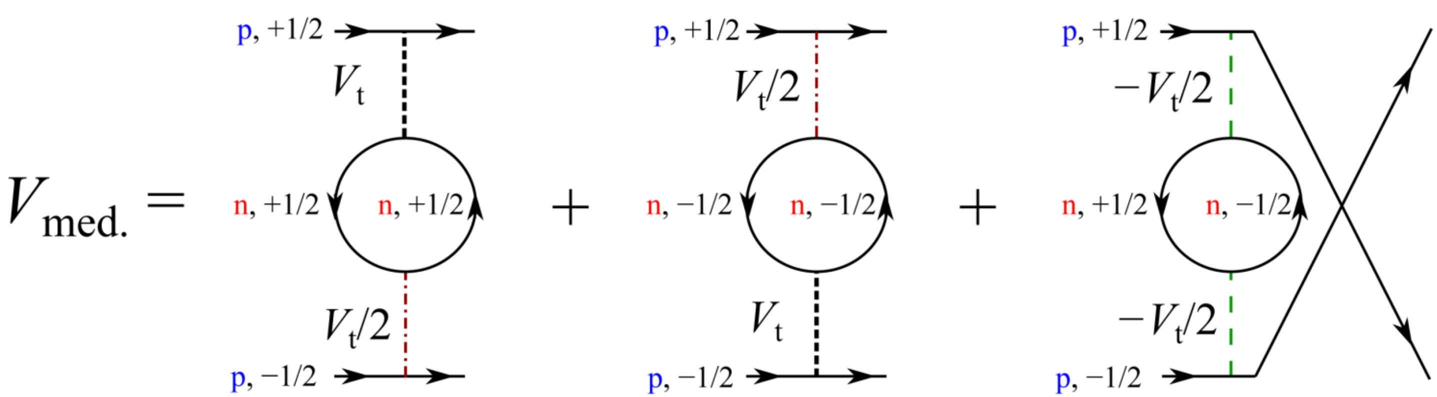




V_{pp}: spin-singlet proton-proton interaction ($a_s = -17.164$ fm)

Close to unitarity (or bound) at subnuclear densities ($\frac{1}{k_{\mathrm{F}}|a_s|} \gg 1$)

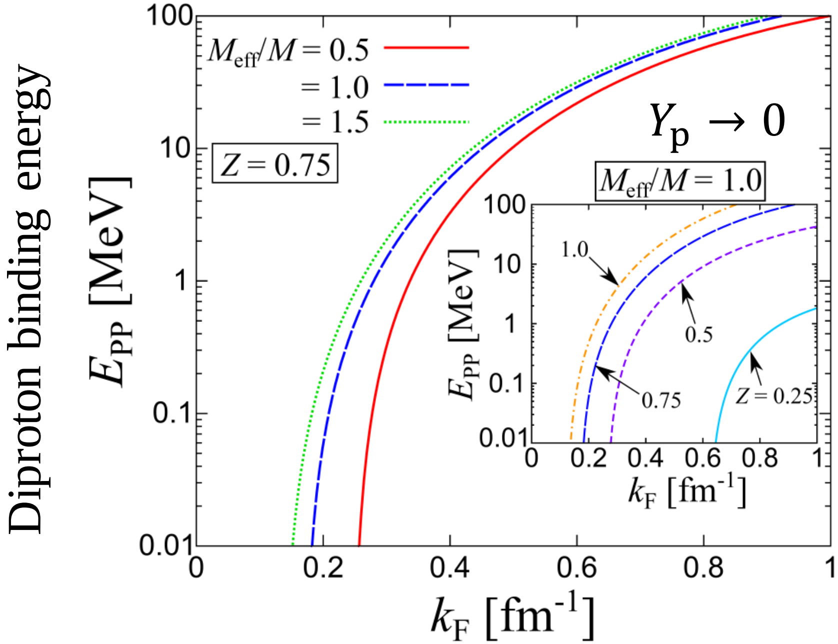
Neutron-mediated proton-proton interaction ($V_{\mathrm{t}} \gg V_{\mathrm{s}} \simeq 0$)



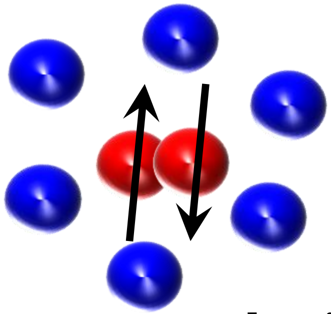
bound diproton induced by neutron density fluctuations

HT, et al., Phys. Lett. B **851**, 138567 (2024).

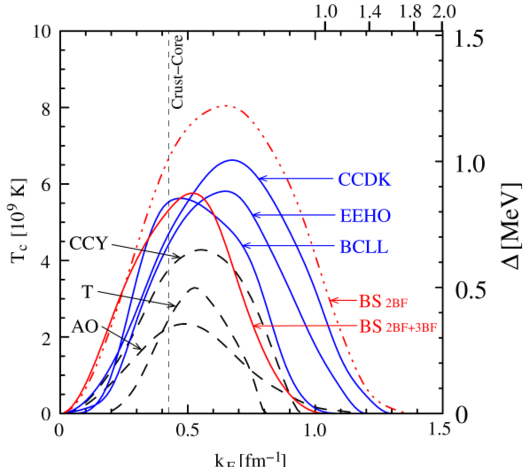
$V_{\mathrm{pp}} + V_{\mathrm{med.}}$ + polaron effect → **bound diproton**



Possibly changing the scenarios of clustering in neutron-rich nuclei and cooling in neutron stars.



***Proton superconductivity**



arXiv:1302.6626

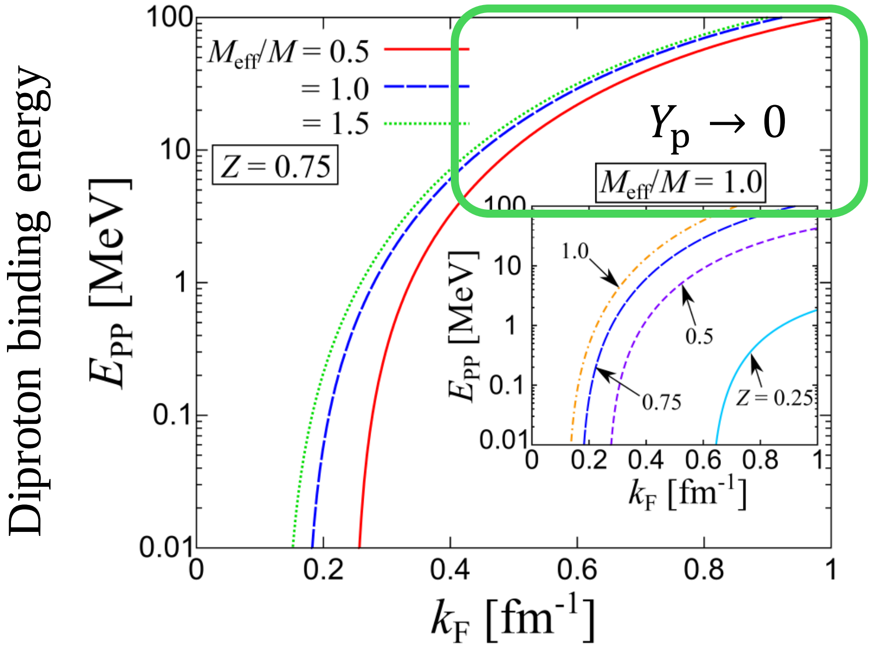
Diproton BEC temperature

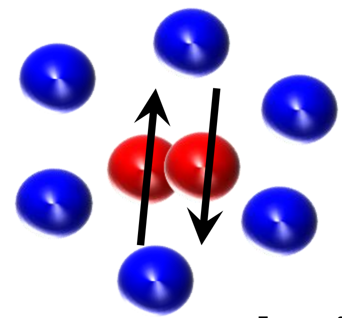
$$T_c^{\mathrm{pp}} = 0.218 \frac{\left(3\pi^2\rho_{\mathrm{p}}\right)^{2/3}}{2M_{\mathrm{eff}}}$$

bound diproton induced by neutron density fluctuations

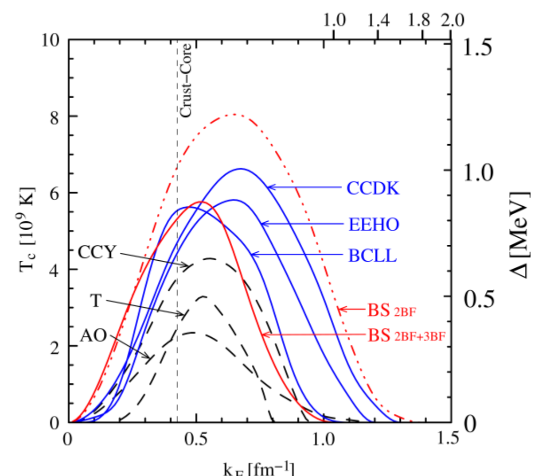
HT, et al., Phys. Lett. B **851**, 138567 (2024).

$V_{\mathrm{pp}} + V_{\mathrm{med.}}$ + polaron effect → **bound diproton**





***Proton superconductivity**



arXiv:1302.6626

Possibly changing the scenarios of clustering in neutron-rich nuclei and cooling in neutron stars.

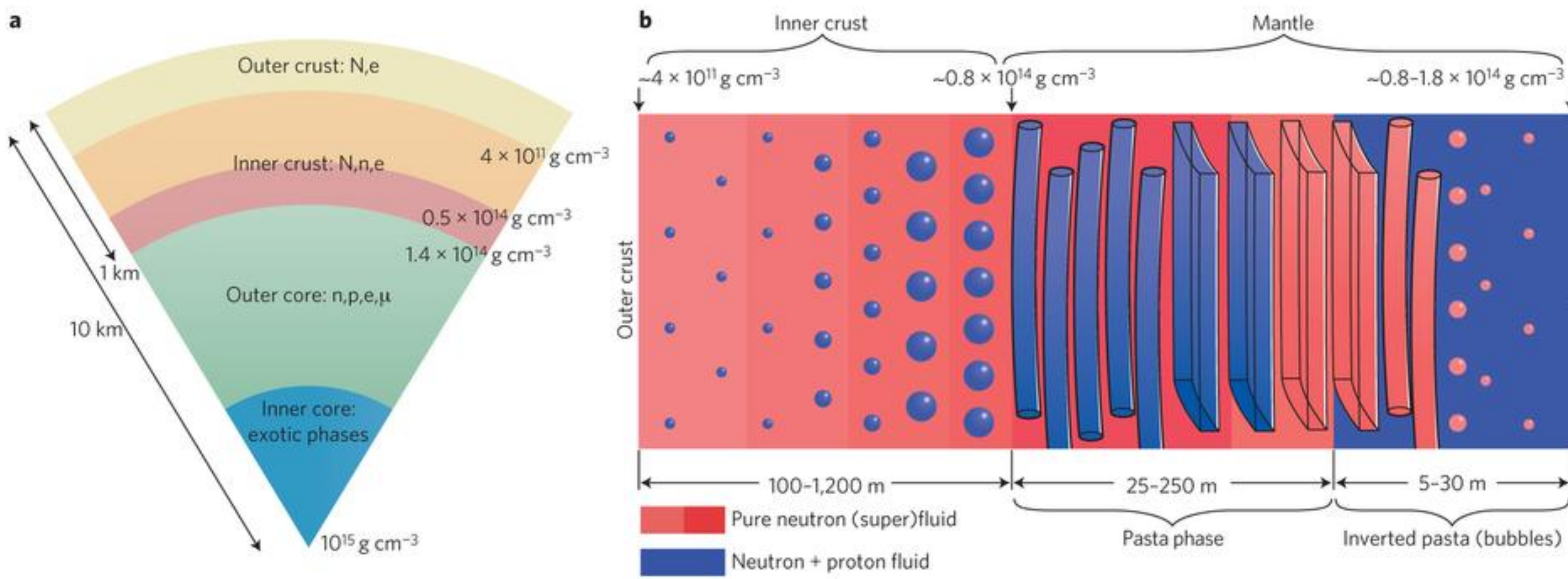
*** But dilute approximation might overestimate E_{pp}**

e.g., lack of medium interaction induced by three-body force
Y. Lim and J. W. Holt, Phys. Rev. C **103**, 025807 (2021)

Diproton BEC temperature

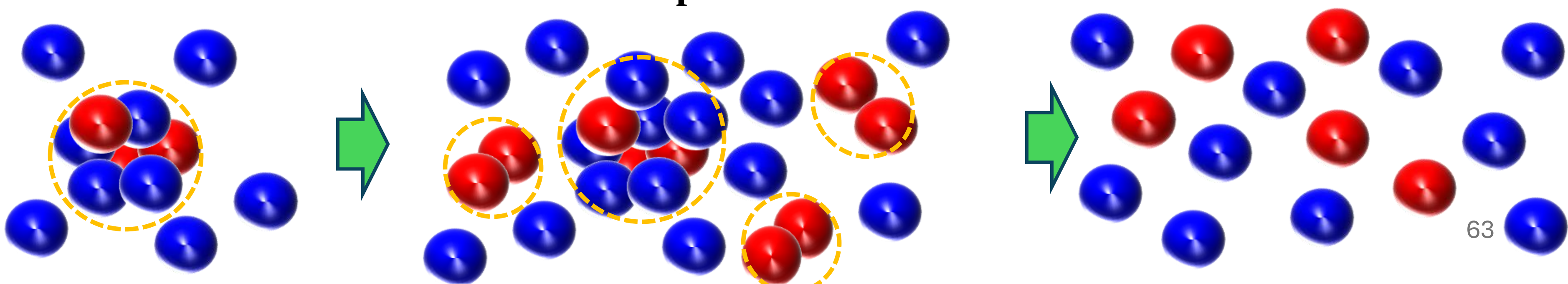
$$T_{\mathrm{c}}^{\mathrm{pp}} = 0.218\,\frac{\left(3\pi^2\rho_{\mathrm{p}}\right)^{2/3}}{2M_{\mathrm{eff}}}$$

Implication of bound protons in neutron-star matter



https://astrobites.org/2017/10/05/nuclear-pasta-in-neutron-stars/

Neutron SF + nuclei **Neutron SF + diproton BEC + nuclei** **Neutron SF + Proton SC**



Outline

- Difference between a ultracold Fermi gas and neutron matter
- Neutron superfluid
- Neutron-proton pairing and proton superconductivity
- Short summary

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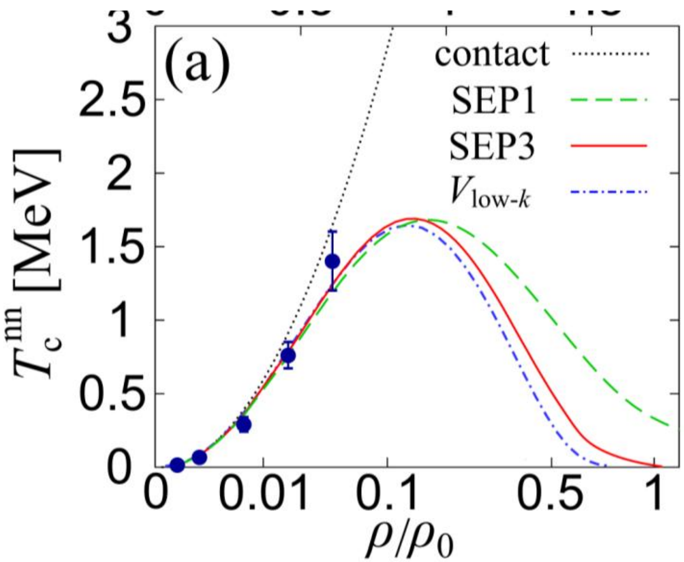
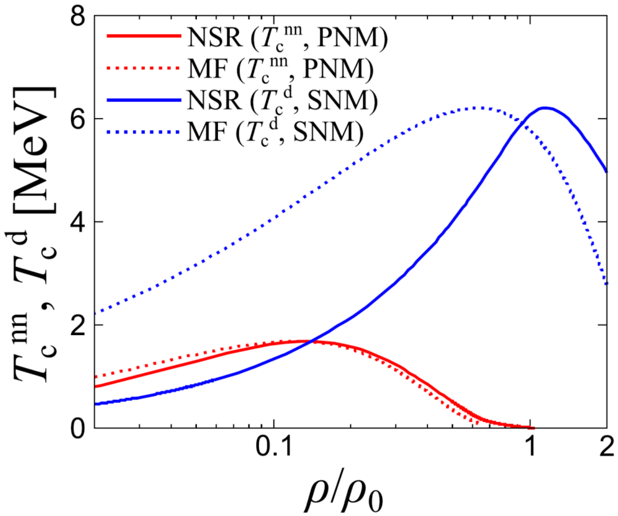
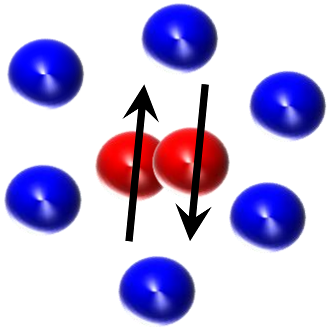
Summary of Part 6

- The difference between a two-component ultracold Fermi gas and neutron star matter has been discussed from a viewpoint of the BCS-BEC crossover theory.
- Strong isovector neutron-proton interaction significantly affects proton superconductivity even without the neutron-proton pairing (deuteron) condensation.
- Proton superconductivity beyond NSR theory has been examined, but more detailed investigation is needed.







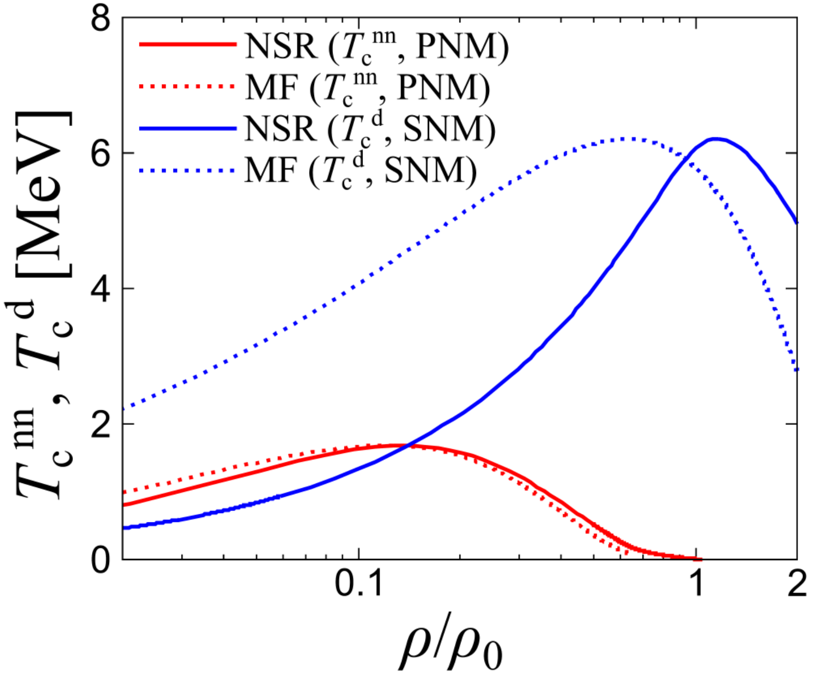
Appendix

Superfluid phase transition beyond BCS

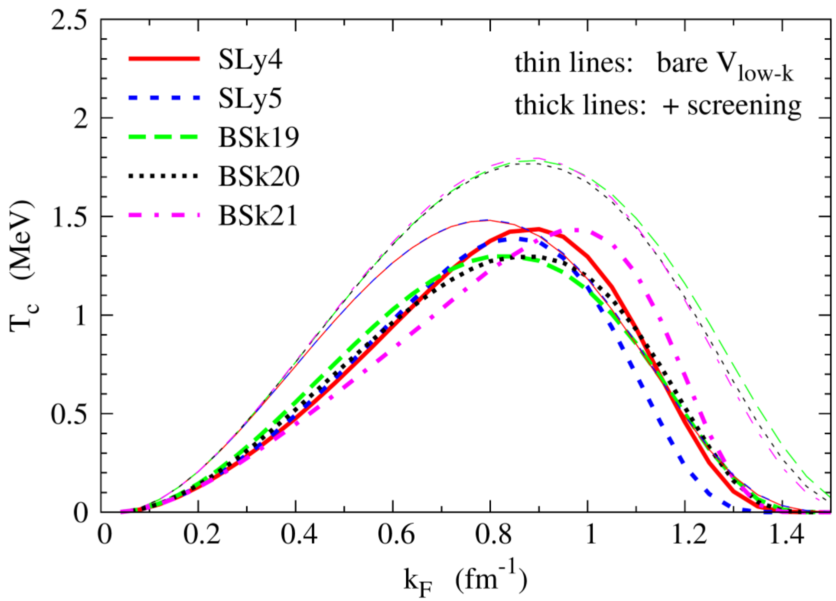
We want to know Neutron density vs. Critical temperature (and Gap)

ρ or k_{F} $\qquad$ T_{c} and Δ

(e.g. NSR correction) $\qquad$ (e.g. GMB correction)



Sci. Rep. **9**, 18477 (2019).

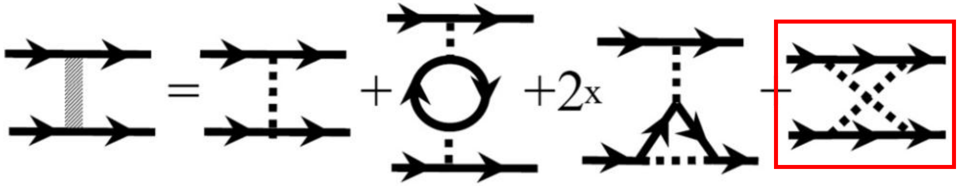


M. Urban *et al.*, PRC **101**, 035803 (2020).

Gor'kov-Melik-Barkhudarov screening

L. P. Gorkov and T. M. Melik-Barkhudarov, Sov. Phys. JETP **13**, 1018 (1961).

Screened attractive interaction by particle-hole fluctuations



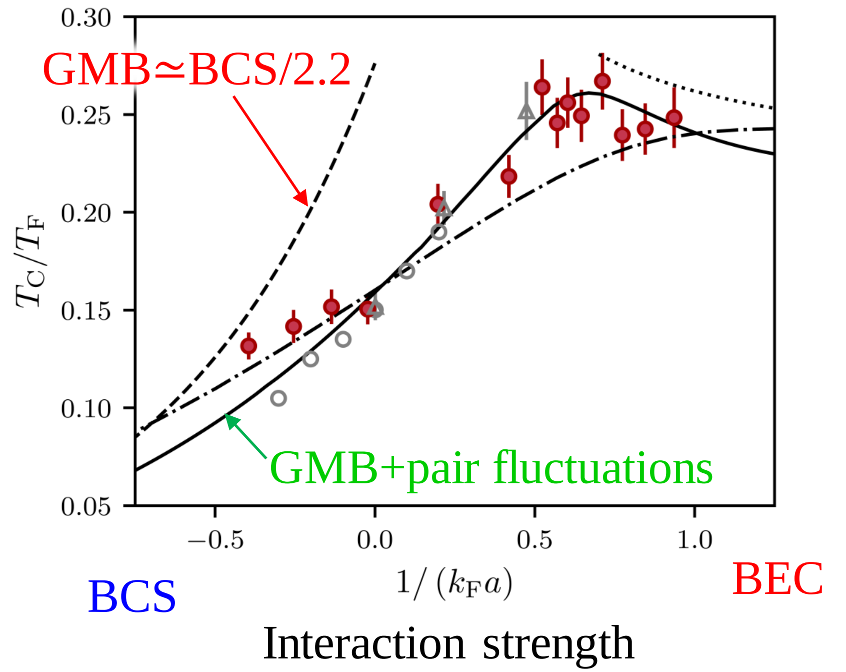
A. Chubukov, et al., PRB **93**, 174516 (2016)

T_c is suppressed by a factor ~**2.2**

$$T_c^{(\mathrm{GMB})} \approx \left(\frac{2}{e}\right)^{7/3} \frac{\gamma}{\pi} T_F e^{\pi/2k_F a_s} \approx 0.28 T_F e^{\pi/2k_F a_s}$$

$$\simeq \frac{1}{(4e)^{1/3}} T_c^{(\mathrm{BCS})} \simeq \frac{1}{2.2} T_c^{(\mathrm{BCS})}$$

Latest experimental and theoretical results of T_c in the BCS-BEC crossover



M. Link, et al., PRL **130**, 203401 (2023).

GMB+pair fluctuations : L. Pisani, A, Perali, P. Pieri, and G. C. Strinati, PRB **97**, 014528 (2018).

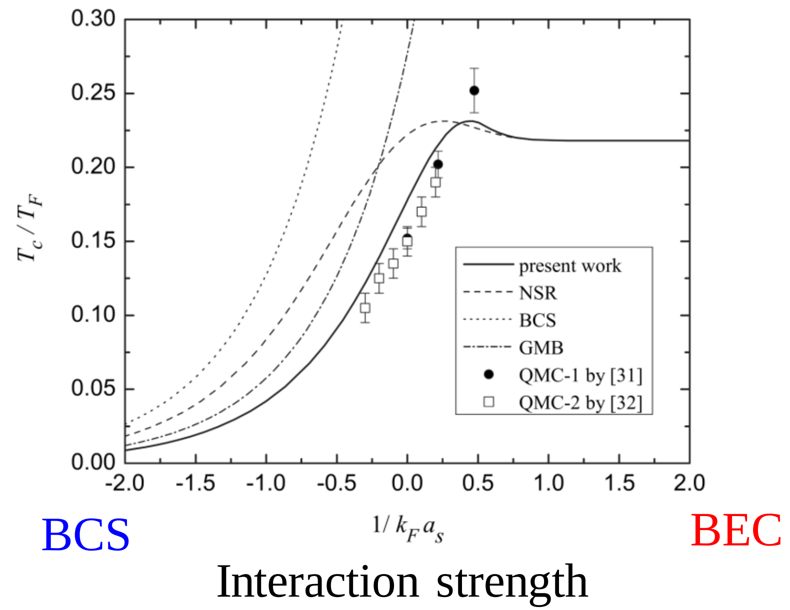
Generalized GMB approach

GMB screening on critical temperature in the single-band BCS-BEC crossover

Z.-Q. Yu, et al., PRA **79**, 053636 (2009).

GMB screening is significant when particle-hole fluctuations are strong.

Large Fermi surface (FS) + finite range



attractive interaction is replaced by the screened interaction

$$U_{\rm tot}(p_1,p_2;p_3,p_4) = g + U_{\rm ind}(p_1,p_2;p_3,p_4) = \frac{g}{1+g\chi(p_1-p_4)}$$

Particle-hole bubble is replaced by the one averaged at the Fermi surface

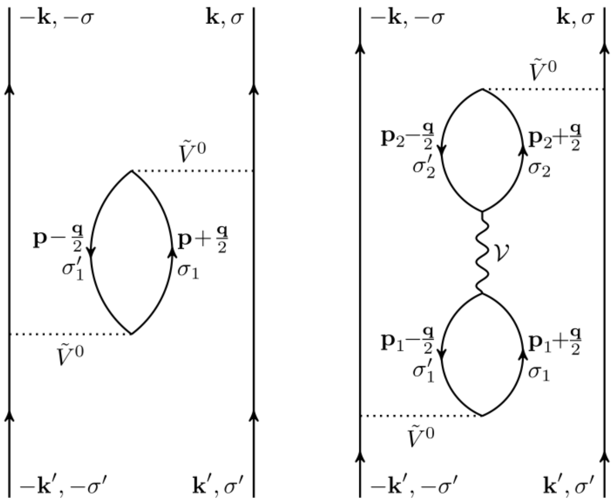
$$\langle\chi\rangle = \frac{m}{4\pi^2\hbar^2}\int_{-1}^{1} d\cos\theta \int_0^\infty dk \frac{k}{k'} f_k \ln\left|\frac{k'-2k}{k'+2k}\right|$$

GMB corrections and beyond

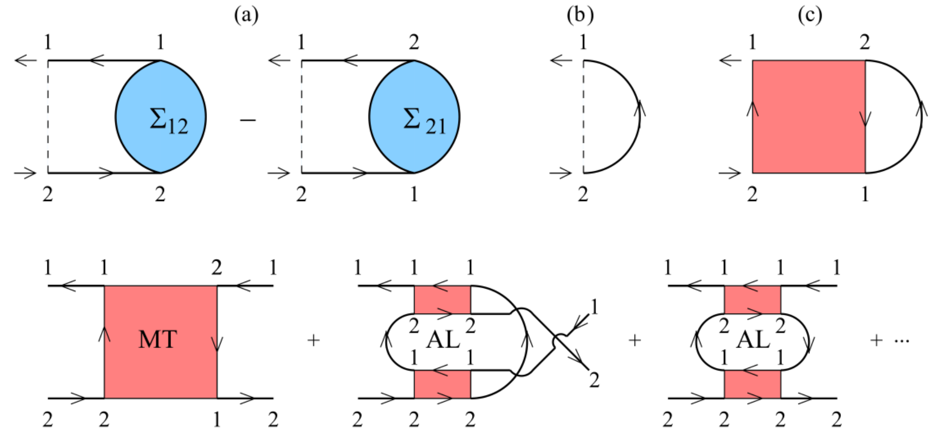
L. G. Cao, U. Lombardo, and P. Schuck, PRC **74**, 064301 (2006).
M. Urban, *et al.*, PRC **101**, 035803 (2020).

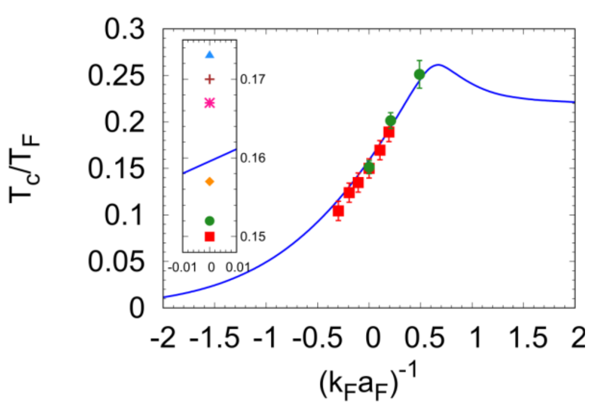
L. Pisani, *et al.*, PRB **97**, 014528 (2018).; **98**, 104507 (2018).

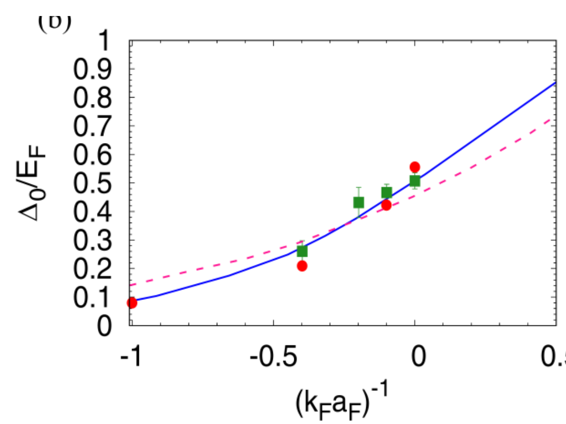
Particle-hole bubbles



$$G^0_{\rm ph}(\mathbf{p},\mathbf{q}) = \frac{n_{\mathbf{p}-\frac{\mathbf{q}}{2}} - n_{\mathbf{p}+\frac{\mathbf{q}}{2}}}{\epsilon_{\mathbf{p}+\frac{\mathbf{q}}{2}} - \epsilon_{\mathbf{p}-\frac{\mathbf{q}}{2}}}$$

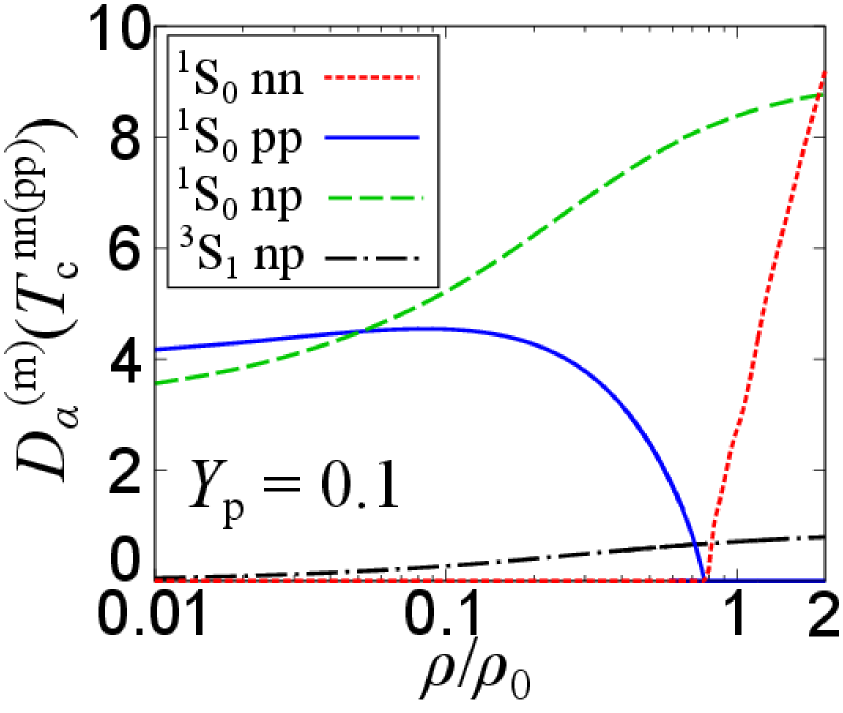




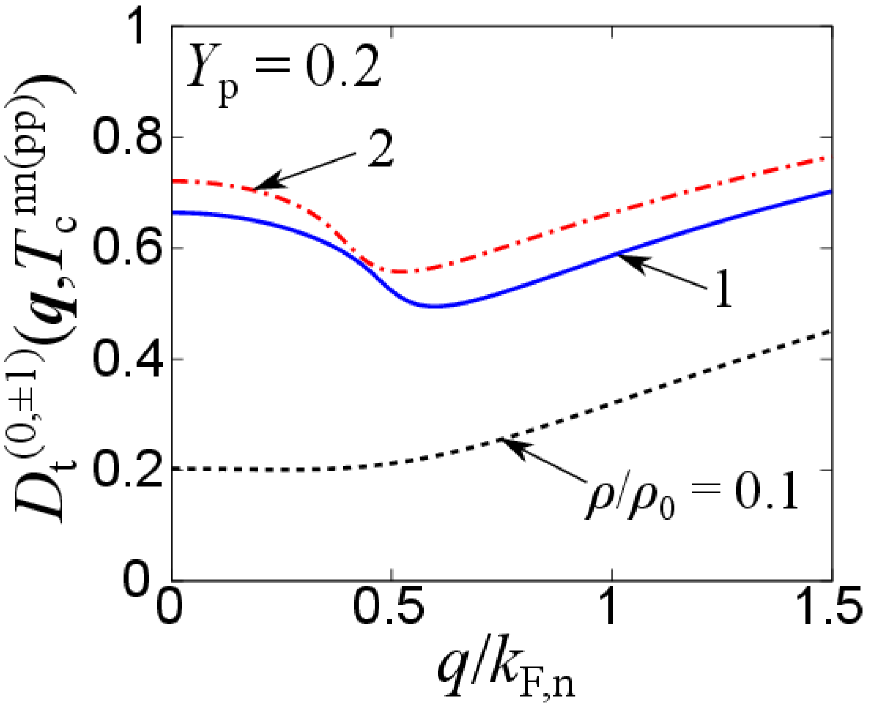


NG mode gap and FFLO-like correlations

▶ NG mode gap at critical temperatures



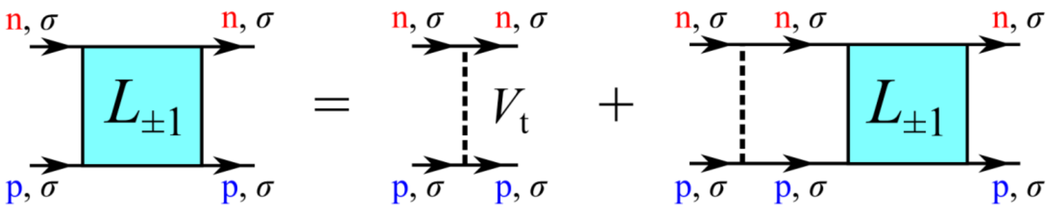
▶ FFLO-like correlations in ANM



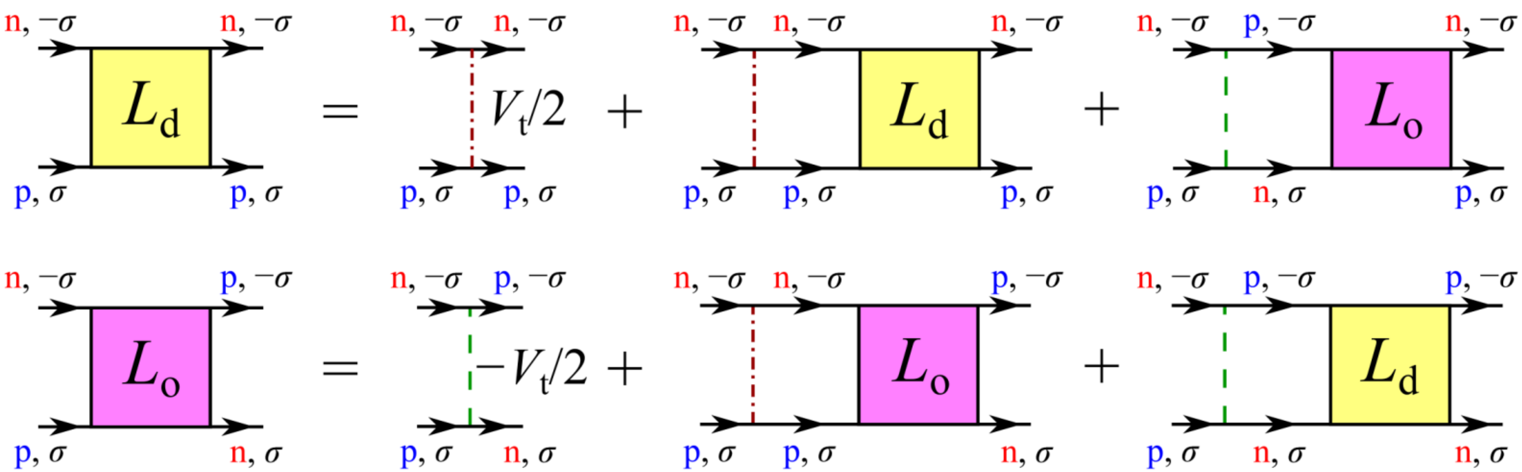
Spin-triplet neutron-proton interaction

$$\Gamma_{\sigma\sigma'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},i\nu_\ell) = \left[L_1(\boldsymbol{q},i\nu_\ell)\delta_{\sigma,\sigma'} + L_0(\boldsymbol{q},i\nu_\ell)\left(1-\delta_{\sigma,\sigma'}\right)\right]\gamma_k\gamma_{k'}$$

(a) $S_z = \pm 1 \quad L_{\pm 1}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},i\nu_\ell) = L_1(\boldsymbol{q},i\nu_\ell)\gamma_k\gamma_{k'}$



(b) $S_z = 0 \quad L_{\rm d}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},i\nu_\ell) = -L_{\rm o}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},i\nu_\ell) \equiv L_0(\boldsymbol{q},i\nu_\ell)\gamma_k\gamma_{k'}$



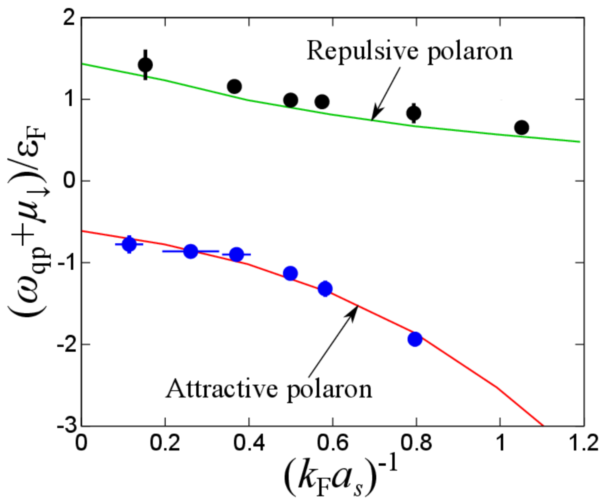
Self-energy of protonic polaron

Many-body T-matrix approach

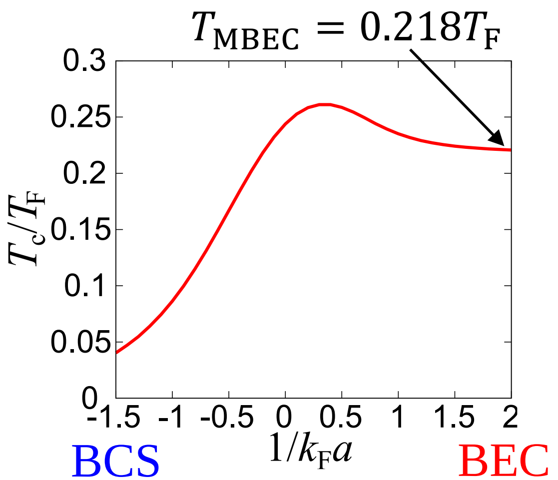
$$\Sigma_{\mathrm{p}\sigma}(\boldsymbol{k}, i\omega_n) = T\sum_{\boldsymbol{q}}\sum_{\sigma'}\sum_{i\nu_\ell}\Gamma_{\sigma\sigma'}(\boldsymbol{q}/2-\boldsymbol{k},\boldsymbol{q}/2-\boldsymbol{k};\boldsymbol{q},i\nu_\ell)G_{\mathrm{n}}(\boldsymbol{q}-\boldsymbol{k},i\nu_\ell-i\omega_n)$$

Reproducing cold atom experiments

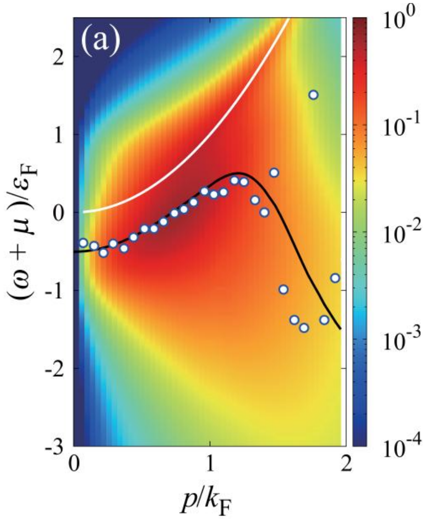
Atomic polaron energy



Critical temperature



Photoemission spectra

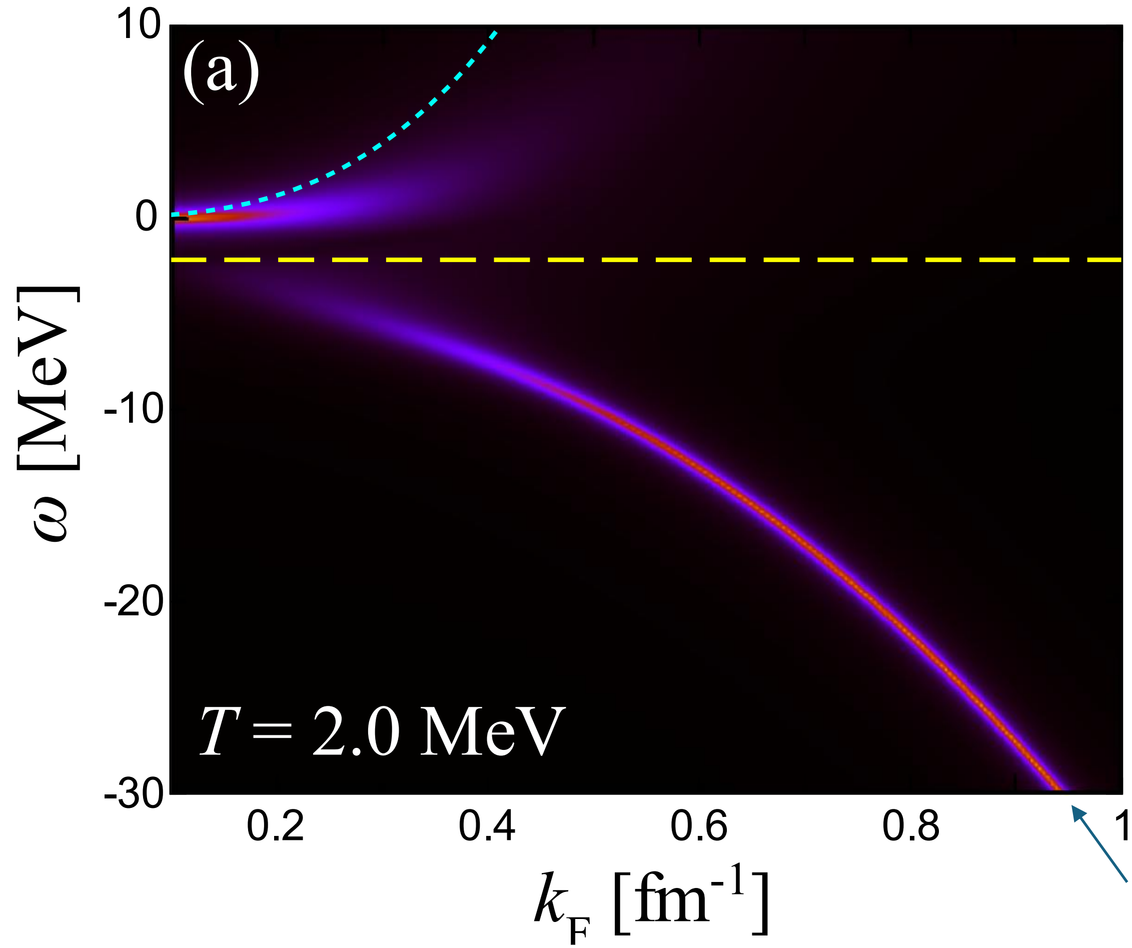


HT and S. Uchino, NJP, **20**, 073048 (2018).

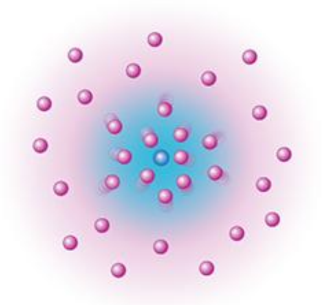
M. Ota, *et al*., PRA **95**,053623 (2017).

Proton spectral weight in neutron-rich matter

$$A_{p\sigma}(\boldsymbol{k}=\boldsymbol{0},\omega) = -\frac{1}{\pi}\mathrm{Im}G_{p\sigma}(\boldsymbol{k}=0,\omega)$$



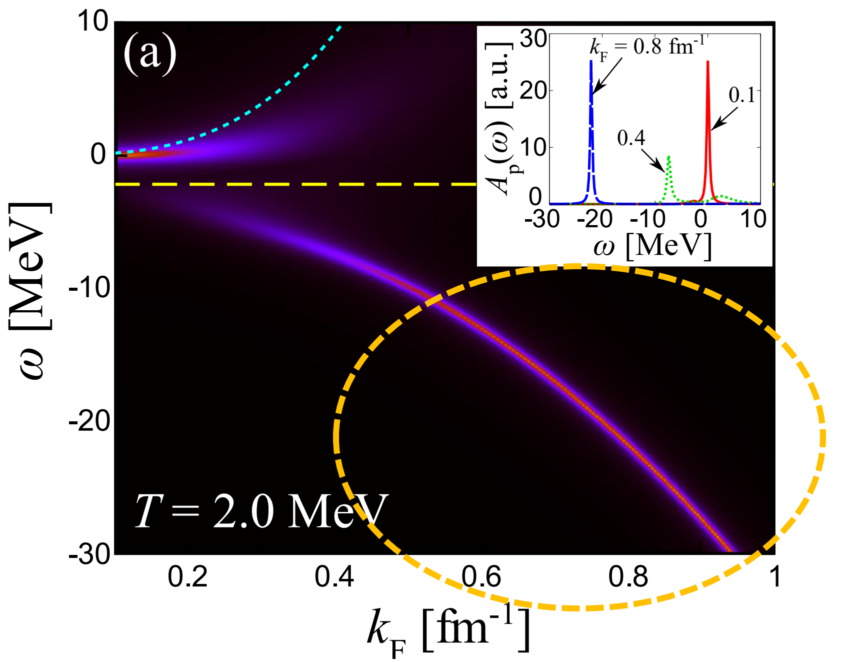
Deuteron energy
~ -2.2 MeV



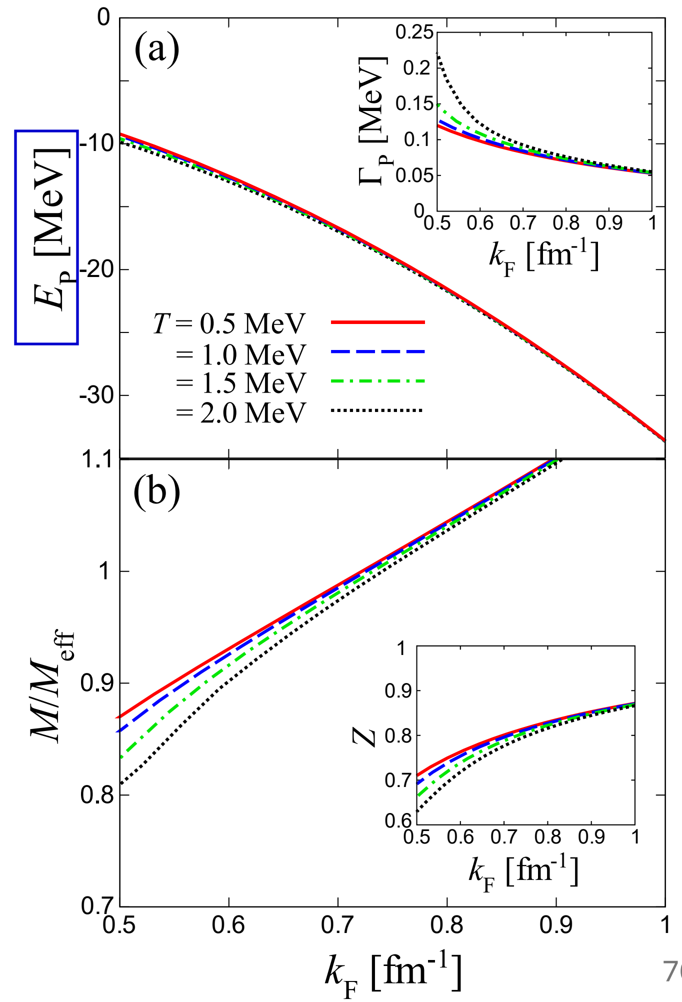
Attractive polaron

Protonic polaron can be stabilized even under the deuteron correlations

Protonic polaron energy



$$G_{p\sigma}(\boldsymbol{k},\omega) \simeq \frac{Z}{\omega_+ - \frac{k^2}{2M_{\rm eff}} - \boxed{E_{\rm P}} + i\Gamma_{\rm P}/2}$$



Landau-Pomeranchuk energy density (T = 0)

$$E/A = E_{\rm PNM} + \boldsymbol{E_{\rm p}}\,\rho_{\rm p}/\rho_{\rm n} + O(\rho_{\rm p}^2)$$

Energy density with symmetry energy

$$E/A = E_{\rm SNM} + \boldsymbol{S}\beta + O(\beta^2) \quad \beta = \frac{\rho_{\rm n} - \rho_{\rm p}}{\rho_{\rm n} + \rho_{\rm p}}$$