Jan. 12-16, 2025

The 2nd Nuclear Physics Tortoise Lecture Series (NPTLS) 2026

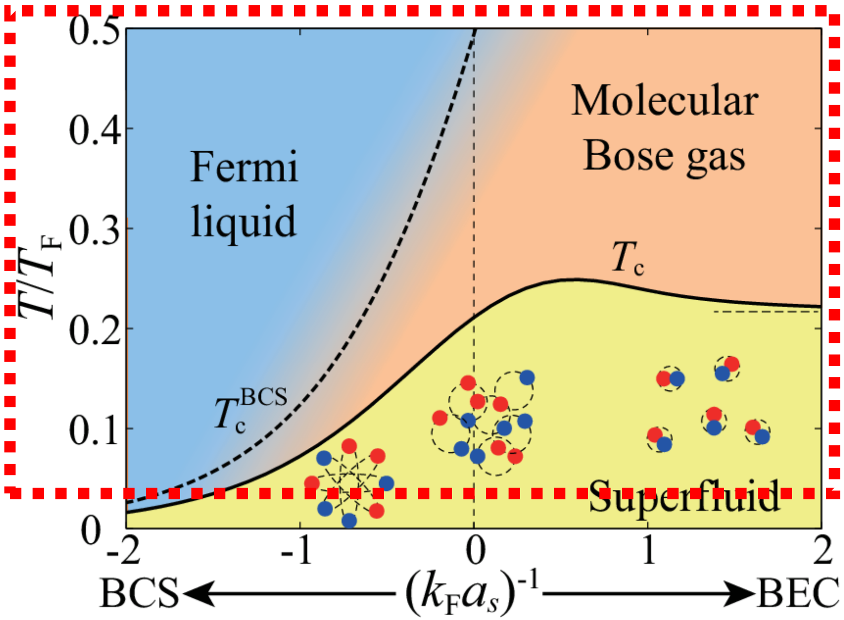
Pairing in ultracold atoms and neutron star

Hiroyuki Tajima

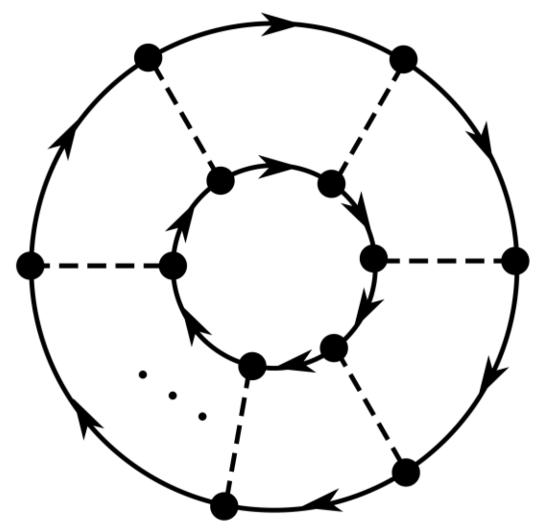
The University of Tokyo, Japan

Part 5
-Pairing fluctuations beyond mean-field theory-

- Understanding the basics of BCS-BEC crossover



Feynman diagram for pairing fluctuations



➡ **Let's try to go beyond mean-field theory at finite temperature**

Outline

- Why beyond mean-field approximation?
- Nozières-Schmitt-Rink theory
- Pseudogap and short-range correlations
- Short summary

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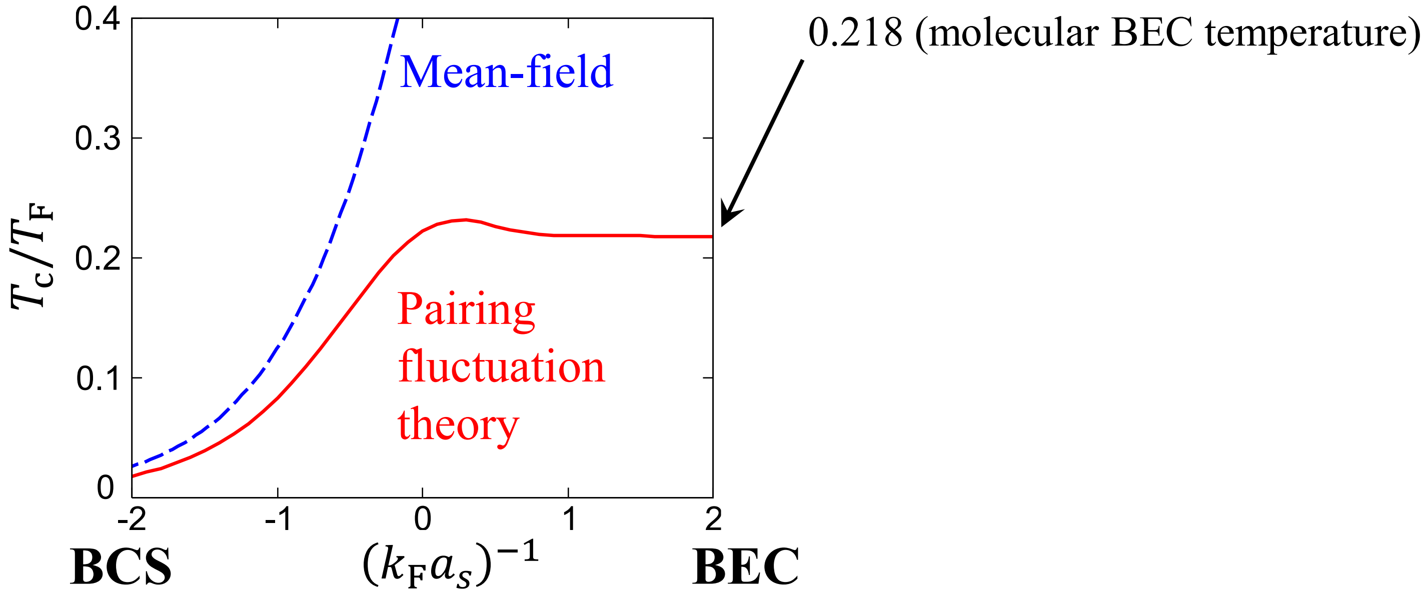
Mean-field theory cannot predict a critical temperature T_c in the strong coupling regime



*$T_F = \varepsilon_F/k_B$: Fermi temperature

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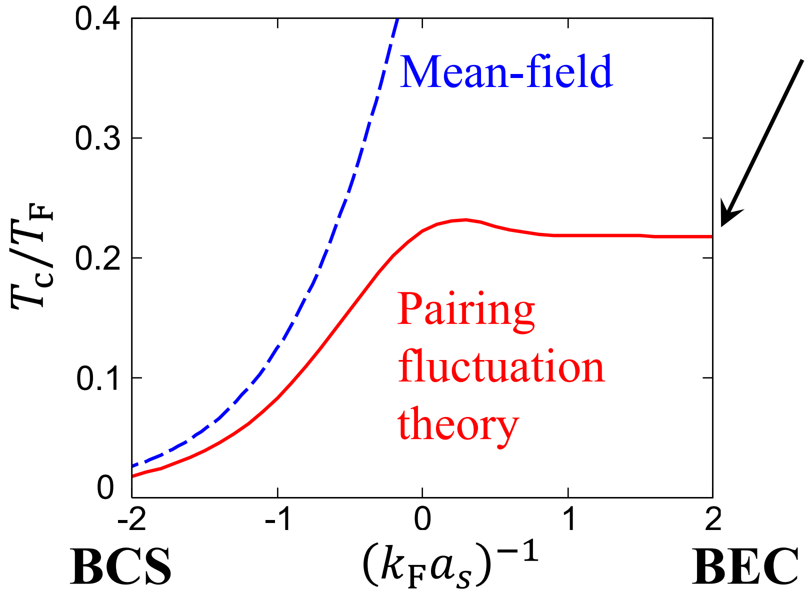

$*T_\mathrm{F} = \varepsilon_\mathrm{F}/k_\mathrm{B}$: Fermi temperature

Fermion number density
= 2 × molecule number density

$$\rho \simeq 2 \times \rho_\mathrm{b} = 2 \sum_{\boldsymbol{P}} \frac{1}{e^{\frac{P^2}{2(2m)T_\mathrm{c}}} - 1}$$

$$= 2 \frac{1}{(2\pi)^3} \int_0^\infty 4\pi P^2 dP \frac{1}{e^{\frac{P^2}{2(2m)T_\mathrm{c}}} - 1}$$

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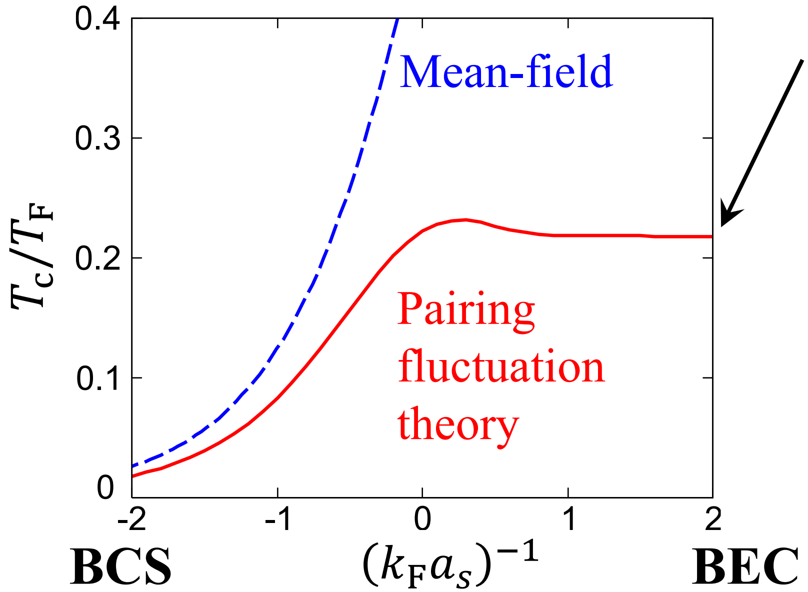

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$$x = P^2/4mT_\mathrm{c} \quad P^2 dP = 4(mT_\mathrm{c})^{3/2}x^{1/2}dx$$

$$= \frac{4(mT_\mathrm{c})^{3/2}}{\pi^2}\int_0^\infty \frac{x^{1/2}dx}{e^x - 1} = 2\left(\frac{mT_\mathrm{c}}{\pi}\right)^{\frac{3}{2}}\zeta(3/2)$$

*$\zeta(x)$: zeta function

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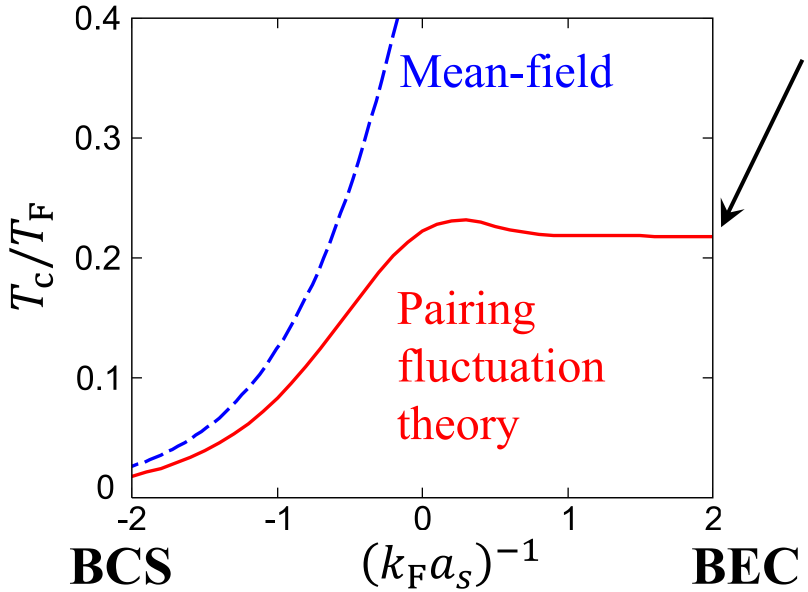

0.218 (molecular BEC temperature)

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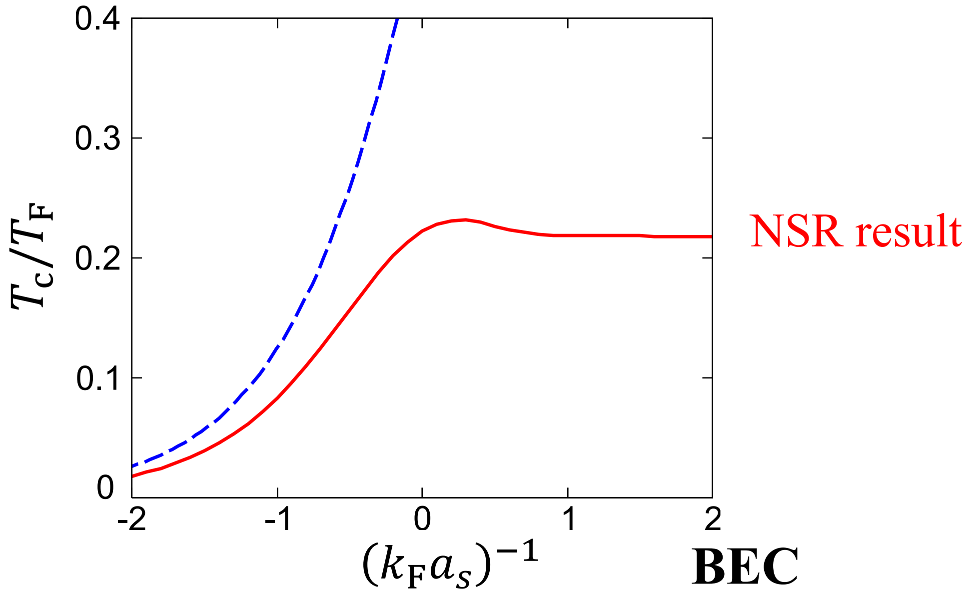
$$\boxed{T_\mathrm{c} \simeq \frac{\pi}{m}\left[\frac{\rho}{2\zeta(3/2)}\right]^{\frac{2}{3}} \simeq 0.218T_\mathrm{F}}$$

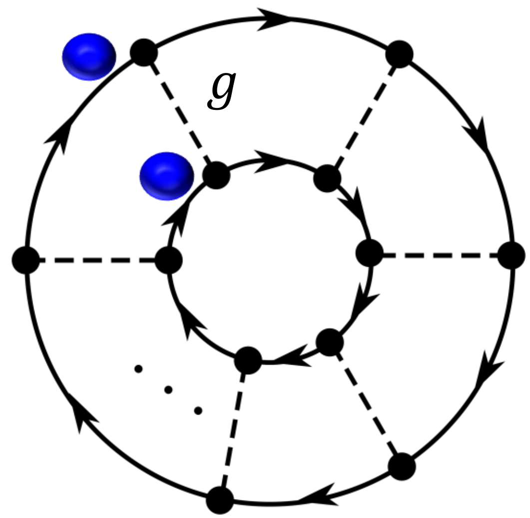
Bose dist. should appear in $\boldsymbol{\rho}$

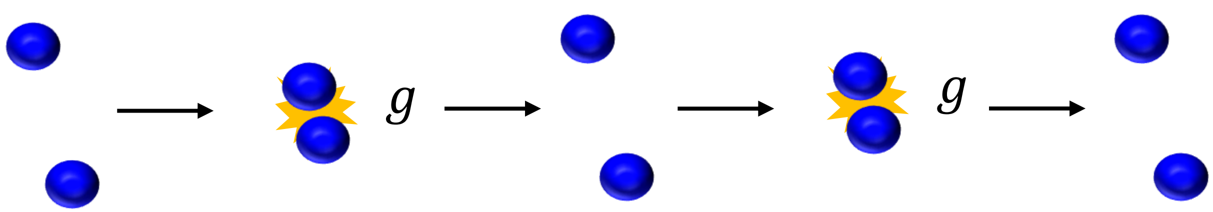
Nozières-Schmitt-Rink theory

P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195–211 (1985).



Pairing fluctuation correction to thermodynamic potential





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Perturbative expansion and thermal Green's function

$$H = \sum_{\boldsymbol{k},\sigma} \xi_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^{\dagger} c_{\boldsymbol{k}\sigma} + g \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} c_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow}^{\dagger} c_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow}^{\dagger} c_{-\boldsymbol{k}'+\boldsymbol{P}/2\downarrow} c_{\boldsymbol{k}'+\boldsymbol{P}/2\uparrow}$$

$$\equiv H_0 + V$$

In contrast to mean-field theory, we keep the P summation, leading to Bose dist.

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Grand-canonical partition function Z and thermodynamic potential Ω

$$Z = \mathrm{Tr}\left[e^{-\beta(H_0+V)}\right] \equiv e^{-\beta\Omega} \qquad * \beta = 1/T$$

At this moment, we know only the non-interacting counterpart $Z_0 = \mathrm{Tr}\left[e^{-\beta H_0}\right] \equiv e^{-\beta\Omega_0}$

➡ **perturbative expansion with respect to *V***

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Real-time perturbation theory

$$|\Psi(t)\rangle = e^{-i(H_0+V)t}|\Psi(0)\rangle$$

$$|\Psi(t)\rangle = e^{-iH_0 t} T_t \exp\left[-i\int dt \hat{V}(t)\right]|\Psi(0)\rangle$$

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Imaginary-time perturbation theory

$$\tau = it \ (0 \le \tau \le \beta)$$

$$\frac{Z}{Z_0} = \left\langle \hat{S}(\beta)\right\rangle_0 \equiv \frac{1}{Z_0}\mathrm{Tr}[e^{-\beta H_0}\hat{S}(\beta)]$$

$$\hat{S}(\beta) = T_\tau \exp\left[-i\int dt \hat{V}(t)\right]$$

T_τ: imaginary time order product

Perturbative expansion of Ω

$$\Omega - \Omega_0 = -\frac{1}{\beta}\ln\left\langle T_\tau \exp\left[-\int_0^\beta d\tau\, \hat{V}(\tau)\right]\right\rangle_0$$

$$= -\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \cdots \int_0^\beta d\tau_n \left\langle T_\tau \hat{V}(\tau_1)\hat{V}(\tau_2)\cdots\hat{V}(\tau_n)\right\rangle_{0\mathrm{c}}$$

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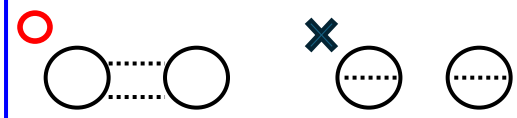
$$\equiv -\frac{1}{\beta}\sum_{n=1}^{\infty}\mathcal{A}_{nc} \quad \text{(connected)}$$

Linked cluster theorem



Only connected

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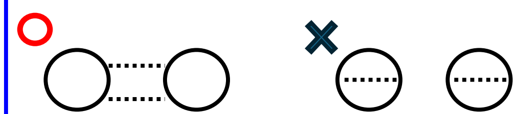

Only connected

Example of linked cluster theorem (2nd order)

$$\Omega_2-\Omega_0=-\frac{1}{\beta}\ln(1+\mathcal{A}_1+\mathcal{A}_2+O(V^3))$$

$$=-\frac{1}{\beta}(\mathcal{A}_1+\mathcal{A}_2)+\frac{1}{2\beta}(\mathcal{A}_1+\mathcal{A}_2)^2+\cdots$$

$$=-\frac{1}{\beta}\left(\mathcal{A}_1+\mathcal{A}_2-\frac{1}{2}\mathcal{A}_1^2\right)+O\left(V^3\right)\equiv-\frac{1}{\beta}(\mathcal{A}_{1c}+\mathcal{A}_{2c})$$

$$\mathcal{A}_1=-\int_0^\beta d\tau_1\left\langle\hat{V}(\tau_1)\right\rangle$$

$$\mathcal{A}_2=\frac{1}{2}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\left\langle T_\tau\hat{V}(\tau_1)\hat{V}(\tau_2)\right\rangle$$

Connected diagram

$$\mathcal{A}_{2c}=\mathcal{A}_2-\frac{1}{2}\mathcal{A}_1^2$$

Perturbative expansion of Ω

$$\boxed{\Omega - \Omega_0 \equiv -\frac{1}{\beta}\sum_{n=1}^{\infty} \mathcal{A}_{nc}}$$

1st-order perturbation (n =1)

$$\mathcal{A}_{1c} = -\int_0^\beta d\tau_1 \left\langle T_\tau \hat{V}(\tau_1)\right\rangle_{0c}$$

$$= -g\int_0^\beta d\tau_1 \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} \left\langle T_\tau c^\dagger_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow}(\tau_1) c^\dagger_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow}(\tau_1) c_{-\boldsymbol{k}'+\boldsymbol{P}/2\downarrow}(\tau_1) c_{\boldsymbol{k}'+\boldsymbol{P}/2\uparrow}(\tau_1)\right\rangle_{0c}$$

Bloch de Dominicis theorem (Wick theorem for $T = 0$)

$$\langle c_1 c_2 c_3 \ \cdots \rangle_{0c} = \sum_{\{\mathcal{P}\}} \mathrm{sgn}(\mathcal{P}) \left\langle c_{1'} c_{2'}\right\rangle_0 \left\langle c_{3'} c_{4'}\right\rangle_0 \cdots$$

$\mathcal{P}$: permutation from $1, 2, \cdots$ to $1', 2', \cdots$

$$\mathcal{A}_{1c} = -g\int_0^\beta d\tau_1 \sum_{\boldsymbol{k},\boldsymbol{P}} G^0_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow}(\tau = -\delta) G^0_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow}(\tau = -\delta)$$

Imaginary-time Green's function: $G^0_{\boldsymbol{k}\sigma}(\tau) = -\left\langle T_\tau c_{\boldsymbol{k}\sigma}(\tau) c^\dagger_{\boldsymbol{k}\sigma}(0)\right\rangle_0$

Perturbative expansion of Ω

1st-order perturbation (*n* =1)

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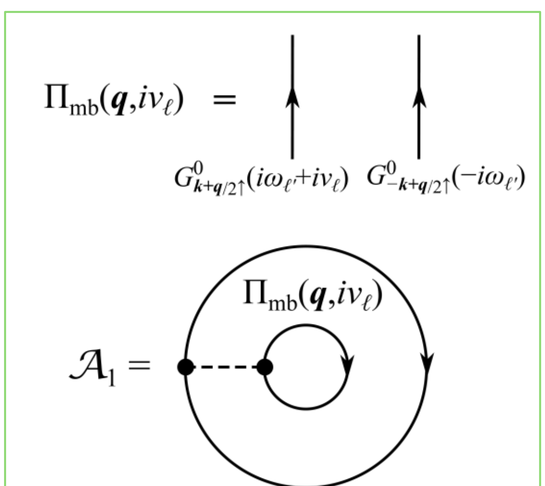

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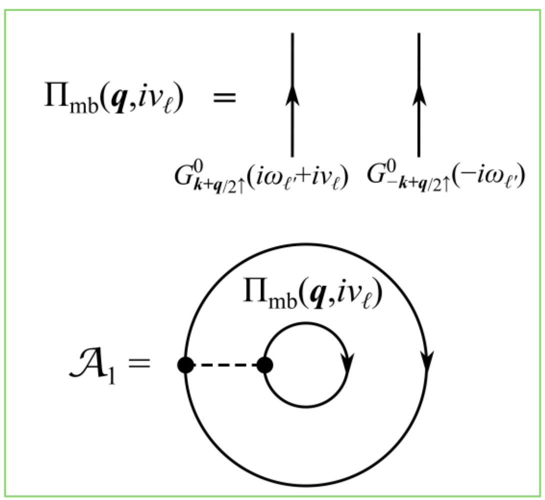
Fermion is anti-periodic for τ

$G^0_{\boldsymbol{k}\sigma}(\tau+\beta) = -G^0_{\boldsymbol{k}\sigma}(\tau)$

Matsubara Green's function (Fourier trans.)

$$G^0_{\boldsymbol{k}\sigma}(i\omega_\ell) = \int_0^\beta d\tau G^0_{\boldsymbol{k}\sigma}(\tau)e^{i\omega_\ell\tau} = \frac{1}{i\omega_\ell - \xi_{\boldsymbol{k}}}$$

$\omega_\ell = (2\ell+1)\pi/\beta$: fermion Matsubara frequency



Equation of motion of $G^0_{\boldsymbol{k}\sigma}(\tau)$

$$\frac{\partial}{\partial \tau} G^0_{\boldsymbol{k}\sigma}(\tau) \equiv -\frac{\partial}{\partial \tau}\left[\theta(\tau)\left\langle c_{\boldsymbol{k}\sigma}(\tau) c^\dagger_{\boldsymbol{k}\sigma}(0)\right\rangle_0 - \theta(-\tau)\left\langle c^\dagger_{\boldsymbol{k}\sigma}(0) c_{\boldsymbol{k}\sigma}(\tau)\right\rangle_0\right]$$

*Heisenberg EOM of $c_{\boldsymbol{k}\sigma}(\tau)$

$$= -\delta(\tau) - \left\langle T_\tau \frac{\partial c_{\boldsymbol{k}\sigma}(\tau)}{\partial \tau} c^\dagger_{\boldsymbol{k}\sigma}(0)\right\rangle_0 = -\delta(\tau) + \xi_{\boldsymbol{k}}\left\langle T_\tau c_{\boldsymbol{k}\sigma}(\tau) c^\dagger_{\boldsymbol{k}\sigma}(0)\right\rangle_0$$

$$\boxed{\Rightarrow \left(\frac{\partial}{\partial \tau} - \xi_{\boldsymbol{k}}\right) G^0_{\boldsymbol{k}\sigma}(\tau) = -\delta(\tau)}$$

$+$

Inverse trans.

$$G^0_{\boldsymbol{k}\sigma}(\tau) = \frac{1}{\beta}\sum_{\ell} G^0_{\boldsymbol{k}\sigma}(i\omega_\ell) e^{-i\omega_\ell \tau}$$

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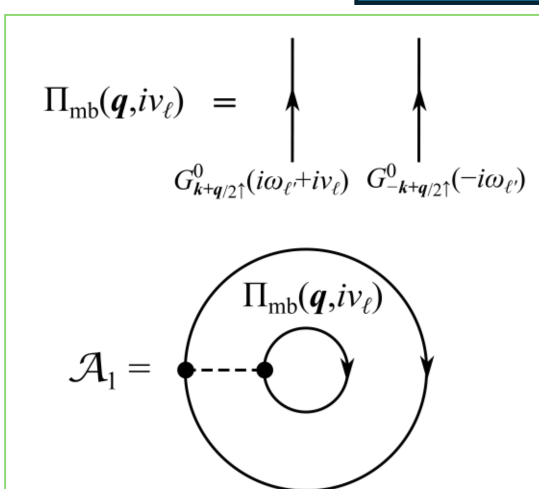

Perturbative expansion of Ω

1st-order perturbation (n =1)

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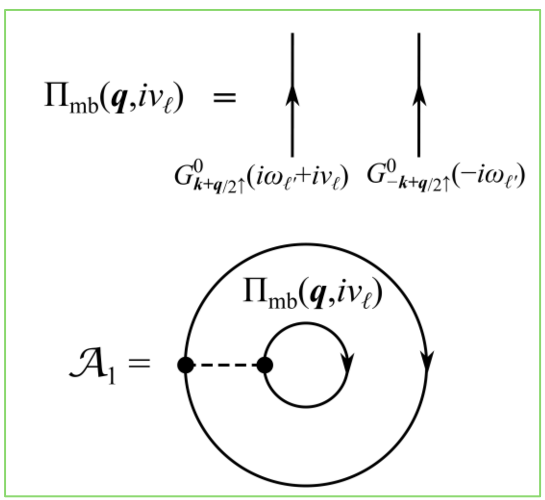

$$\mathcal{A}_{1c} = -\sum_{\boldsymbol{P},\ell} g\,\Pi_{\rm mb}(\boldsymbol{P},i\nu_\ell)e^{i\nu_\ell\delta}$$

$\nu_\ell = 2\ell\pi/\beta$: boson Matsubara frequency

Pair propagator

$$\Pi_{\rm mb}(\boldsymbol{P},i\nu_\ell) = \frac{1}{\beta}\sum_{\boldsymbol{k},\ell} G^0_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow}(i\omega_{\ell'}+i\nu_\ell)G^0_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow}(-i\omega_{\ell'})$$

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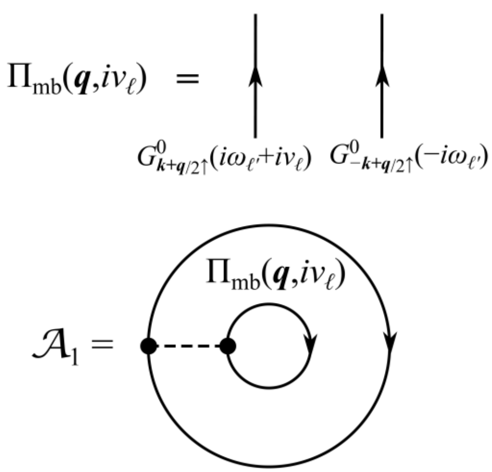

$$\mathcal{A}_{1\rm c} = \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} g\,\Pi_{\rm mb}(\boldsymbol{P}, i\nu_\ell)e^{i\nu_\ell\delta}$$

Pair propagator

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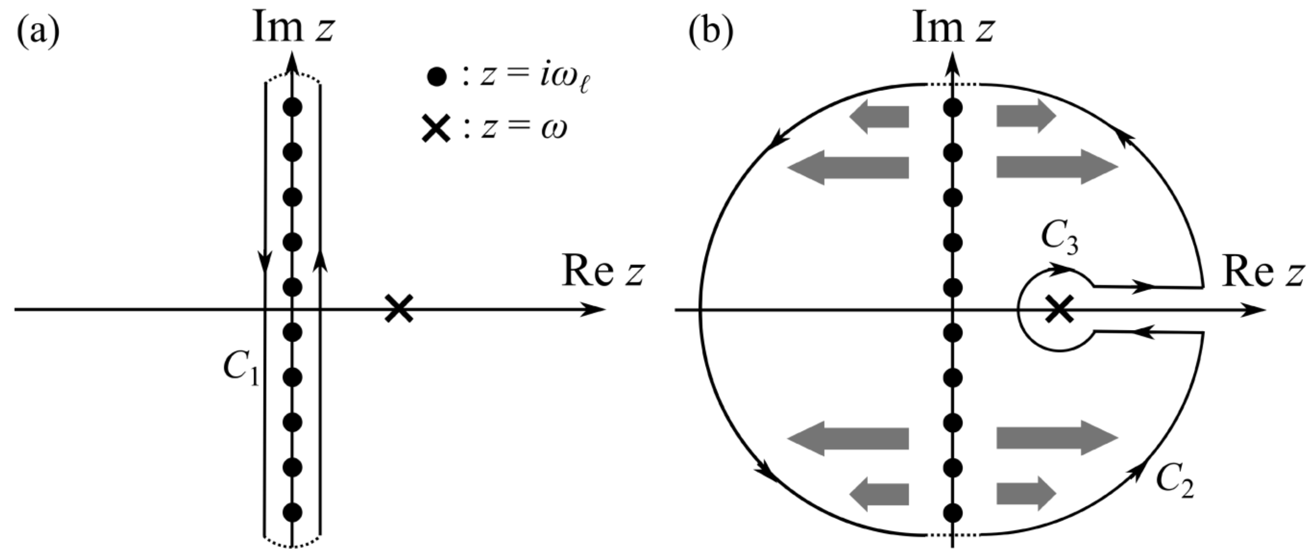
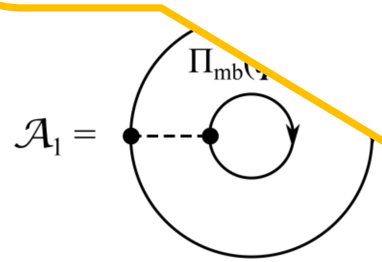
# of Ω





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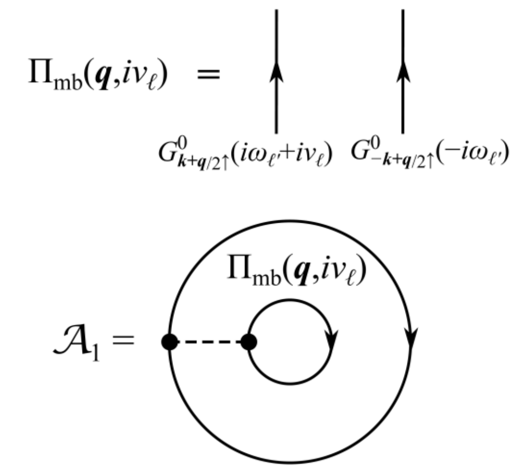
$$= \sum_{\boldsymbol{k}} \oint_C \frac{f(z)dz}{2\pi i} \frac{1}{(z + i\nu_\ell - \xi_{\boldsymbol{k}+\boldsymbol{P}/2})(z + \xi_{-\boldsymbol{k}+\boldsymbol{P}/2})}$$

$$= -\sum_{\boldsymbol{k}} \frac{1 - f(\xi_{\boldsymbol{k}+\boldsymbol{P}/2}) - f(\xi_{-\boldsymbol{k}+\boldsymbol{P}/2})}{i\nu_\ell - \xi_{\boldsymbol{k}+\boldsymbol{P}/2} - \xi_{-\boldsymbol{k}+\boldsymbol{P}/2}}$$

Perturbative expansion of Ω

1st-order perturbation (n =1)



$$\mathcal{A}_{1c} = \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} g\,\Pi_{\rm mb}(\boldsymbol{P}, i\nu_\ell) e^{i\nu_\ell\delta}$$

Pair propagator

$$\Pi_{\rm mb}(\boldsymbol{P}, i\nu_\ell) = \frac{1}{\beta}\sum_{\boldsymbol{k},\ell} G^0_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow}(i\omega_{\ell'} + i\nu_\ell) G^0_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow}(-i\omega_{\ell'})$$

$$= \frac{1}{\beta}\sum_{\boldsymbol{k},\ell} \frac{1}{(i\omega_{\ell'} + i\nu_\ell - \xi_{\boldsymbol{k}+\boldsymbol{P}/2})(-i\omega_{\ell'} - \xi_{-\boldsymbol{k}+\boldsymbol{P}/2})}$$

$$= \sum_{\boldsymbol{k}} \oint_C \frac{f(z)dz}{2\pi i} \frac{1}{(z + i\nu_\ell - \xi_{\boldsymbol{k}+\boldsymbol{P}/2})(z + \xi_{-\boldsymbol{k}+\boldsymbol{P}/2})}$$

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Pauli block factor

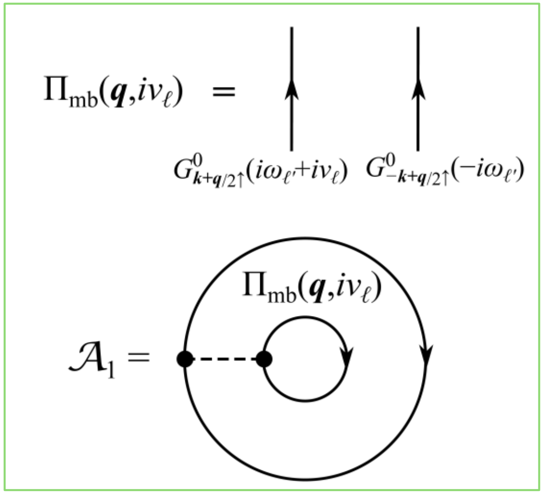
$$1 - f(\xi_{\boldsymbol{k}+\boldsymbol{P}/2}) - f(\xi_{-\boldsymbol{k}+\boldsymbol{P}/2})$$

$$= \mathbf{pp} - \mathbf{hh}$$

$$[1 - f(\xi_{\boldsymbol{k}+\boldsymbol{P}/2})][1 - f(\xi_{-\boldsymbol{k}+\boldsymbol{P}/2})]$$
$$- f(\xi_{\boldsymbol{k}+\boldsymbol{P}/2}) f(\xi_{-\boldsymbol{k}+\boldsymbol{P}/2})$$

Perturbative expansion of Ω

1st-order perturbation (n =1)



$$\mathcal{A}_{1\rm c} = \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} g\,\Pi_{\rm mb}(\boldsymbol{P},i\nu_\ell)e^{i\nu_\ell\delta}$$

Pair propagator

$$\Pi_{\rm mb}(\boldsymbol{P},i\nu_\ell) = \frac{1}{\beta}\sum_{\boldsymbol{k},\ell} G^0_{\boldsymbol{k}+\boldsymbol{P}/2\uparrow}(i\omega_{\ell'}+i\nu_\ell)G^0_{-\boldsymbol{k}+\boldsymbol{P}/2\downarrow}(-i\omega_{\ell'})$$

$$= \frac{1}{\beta}\sum_{\boldsymbol{k},\ell} \frac{1}{(i\omega_{\ell'}+i\nu_\ell-\xi_{\boldsymbol{k}+\boldsymbol{P}/2})(-i\omega_{\ell'}-\xi_{-\boldsymbol{k}+\boldsymbol{P}/2})}$$

$$= \sum_{\boldsymbol{k}}\oint_C \frac{f(z)dz}{2\pi i}\frac{1}{(z+i\nu_\ell-\xi_{\boldsymbol{k}+\boldsymbol{P}/2})(z+\xi_{-\boldsymbol{k}+\boldsymbol{P}/2})}$$

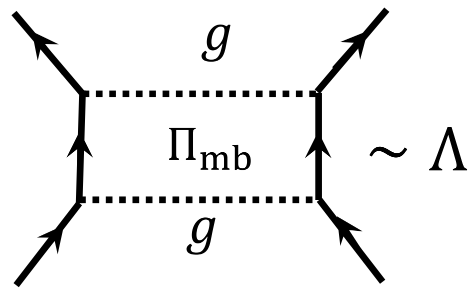
$$= -\sum_{\boldsymbol{k}} \frac{1-f(\xi_{\boldsymbol{k}+\boldsymbol{P}/2})-f(\xi_{-\boldsymbol{k}+\boldsymbol{P}/2})}{i\nu_\ell-\xi_{\boldsymbol{k}+\boldsymbol{P}/2}-\xi_{-\boldsymbol{k}+\boldsymbol{P}/2}}$$

$$\boxed{\Pi_{\rm mb}(\boldsymbol{P}=\boldsymbol{0},i\nu_\ell=0) = \sum_{\boldsymbol{k}}\frac{1-2f(\xi_{\boldsymbol{k}})}{2\xi_{\boldsymbol{k}}} \xrightarrow{T=\mu=0} \sum_{\boldsymbol{k}}\frac{m}{k^2}}$$

UV divergence!

Ladder approximation

It is found that pair propagator Π_{mb} exhibits an UV divergence



Since we cannot sum up all the diagrams, we focus on the ladder-type ones,
Because the UV divergence implies its importance

Ladder approximation

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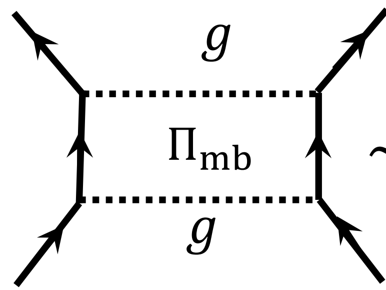
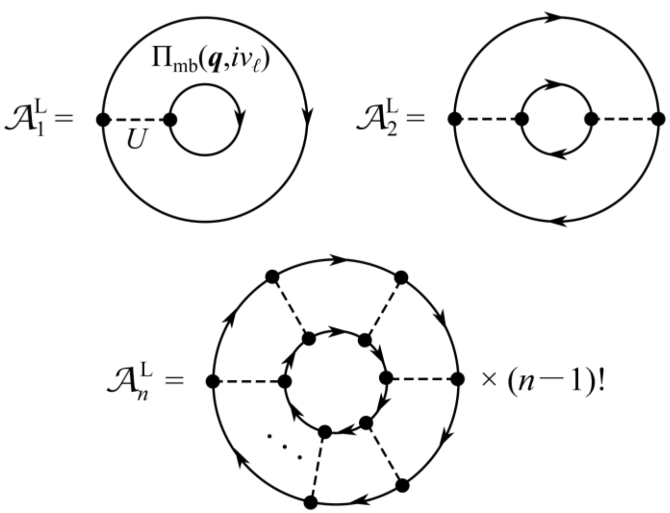


$\sim \Lambda$

Since we cannot sum up all the diagrams, we focus on the ladder-type ones,
Because the UV divergence implies its importance

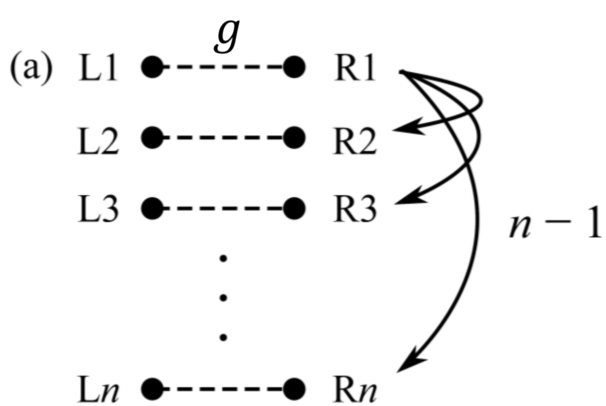
NSR thermodynamic potential



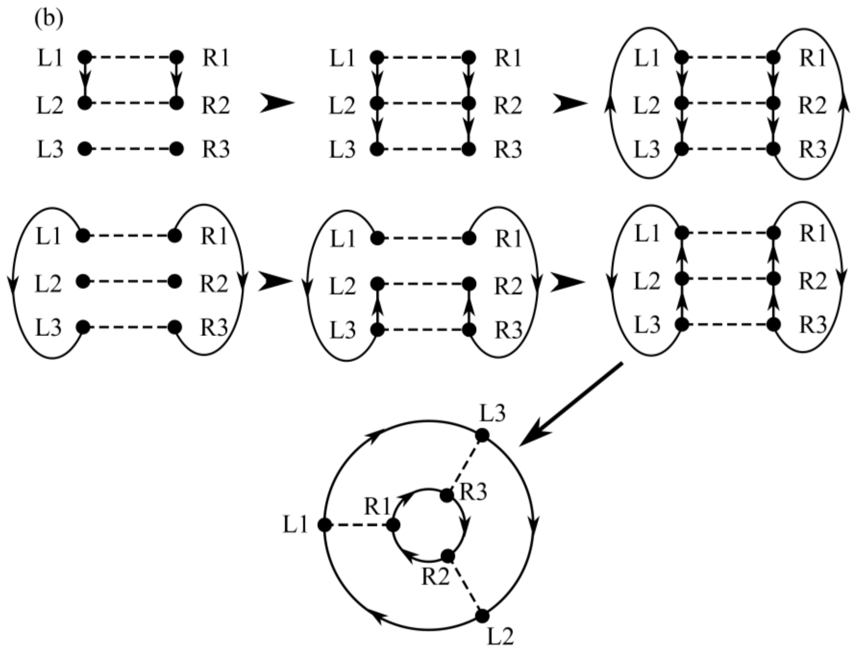
$$
\begin{aligned}
\Omega_{\mathrm{NSR}} - \Omega_0 &= -\frac{1}{\beta}\sum_{n=1}^{\infty} \mathcal{A}_n^{\mathrm{L}} \\
&= \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{\boldsymbol{q}}\sum_{\ell}\left[U\Pi_{\mathrm{mb}}(\boldsymbol{q}, i\nu_\ell)\right]^n e^{i\nu_\ell\eta} \times (n-1)! \\
&= \frac{1}{\beta}\sum_{n=1}^{\infty}\sum_{\boldsymbol{q}}\sum_{\ell}\frac{1}{n}\left[-U\Pi_{\mathrm{mb}}(\boldsymbol{q}, i\nu_\ell)\right]^n e^{i\nu_\ell\eta} \\
&= \frac{1}{\beta}\sum_{\boldsymbol{q}}\sum_{\ell}\ln\left[1 + U\Pi_{\mathrm{mb}}(\boldsymbol{q}, i\nu_\ell)\right] e^{i\nu_\ell\eta}, \qquad (
\end{aligned}
$$

$* \ U = g, \boldsymbol{q} = \boldsymbol{P}$

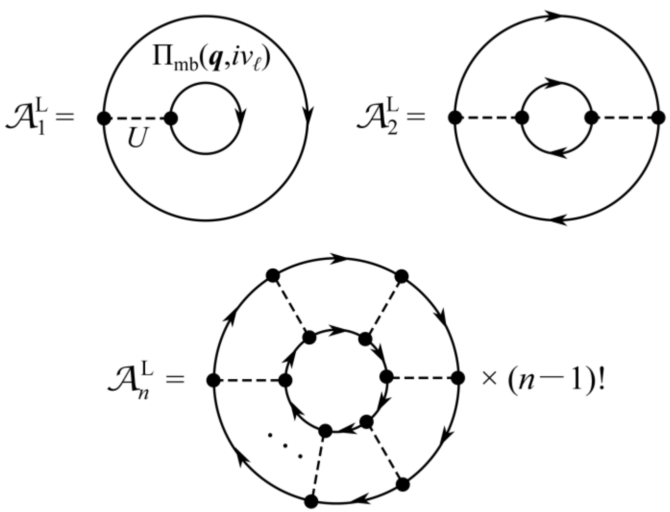
Why $(n-1)$?



Example of $n = 3$



NSR thermodynamic po



$$\Omega_{\mathrm{NSR}} - \Omega_0 = -\frac{1}{\beta}\sum_{n=1}^{\infty}\mathcal{A}_n^{\mathrm{L}}$$

$$= \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{\boldsymbol{q}}\sum_{\ell}\left[U\Pi_{\mathrm{mb}}(\boldsymbol{q}, i\nu_\ell)\right]^n e^{i\nu_\ell\eta} \times (n-1)!$$

$$= \frac{1}{\beta}\sum_{n=1}^{\infty}\sum_{\boldsymbol{q}}\sum_{\ell}\frac{1}{n}\left[-U\Pi_{\mathrm{mb}}(\boldsymbol{q}, i\nu_\ell)\right]^n e^{i\nu_\ell\eta}$$

$$= \frac{1}{\beta}\sum_{\boldsymbol{q}}\sum_{\ell}\ln\left[1 + U\Pi_{\mathrm{mb}}(\boldsymbol{q}, i\nu_\ell)\right] e^{i\nu_\ell\eta},$$

$*\ U = g, \boldsymbol{q} = \boldsymbol{P}$

NSR theory with renormalization

$$\Omega_{\mathrm{NSR}} - \Omega_0 = \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} \ln[1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)]\, e^{i\nu_\ell\delta}$$

Renormalization relation

$$\boxed{\frac{1}{g} = \frac{m}{4\pi a_s} - \frac{m\Lambda}{2\pi^2} \equiv \frac{m}{4\pi a_s} - \sum_{|\boldsymbol{k}|\leq\Lambda} \frac{m}{k^2}}$$

Regularized NSR thermodynamic potential

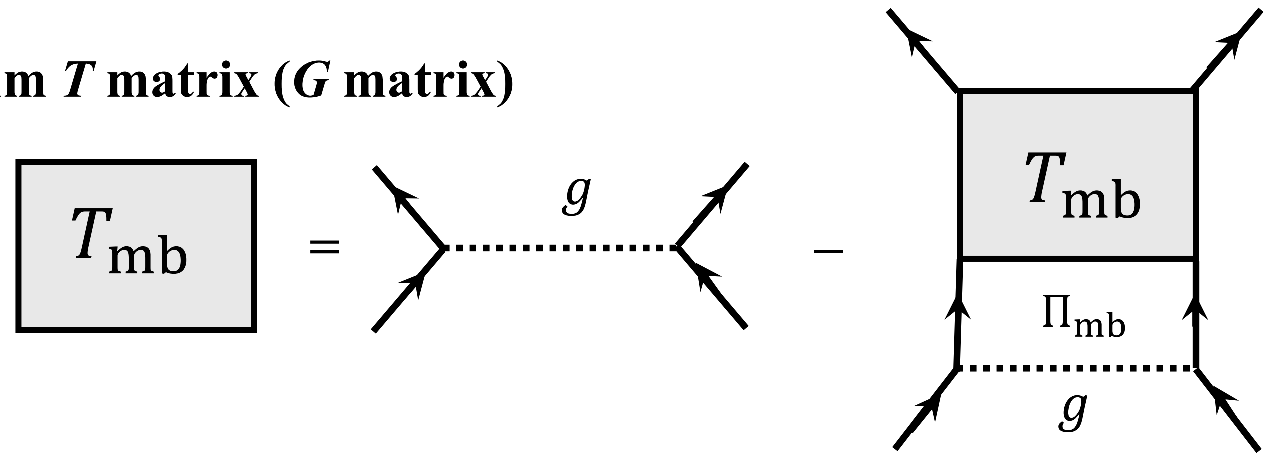
$$\boxed{\Omega_{\mathrm{NSR}} - \Omega_0 = \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} \ln\left[1 + \frac{4\pi a_s}{m}\left\{\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) - \sum_{\boldsymbol{k}} \frac{m}{k^2}\right\}\right] e^{i\nu_\ell\delta}}$$

Let's check the number density: $\rho_{\mathrm{NSR}} = -\frac{\partial\Omega_0}{\partial\mu} - \frac{\partial\Omega_{\mathrm{NSR}}}{\partial\mu}$

$$\rho_{\mathrm{NSR}} = 2\sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) - \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} \boxed{\frac{\frac{4\pi a_s}{m}}{1+\frac{4\pi a_s}{m}\left\{\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) - \sum_{\boldsymbol{k}}\frac{m}{k^2}\right\}}} \frac{\partial \Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}{\partial \mu} e^{i\nu\ell\delta}$$

$$T_{mb}(\boldsymbol{P}, i\nu_\ell) = \frac{1}{\frac{m}{4\pi a_s} + \Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) - \sum_{\boldsymbol{k}}\frac{m}{k^2}} \equiv \frac{g}{1 + g\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}$$

In-medium *T* matrix (*G* matrix)



* In the *G*-matrix case for nuclear case, the hh part is neglected for simplicity

$$\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) = -\sum_{\boldsymbol{k}} \frac{[1 - f(\xi_{\boldsymbol{k}+\boldsymbol{P}/2})][1 - f(\xi_{-\boldsymbol{k}+\boldsymbol{P}/2})] - \cancel{f(\xi_{\boldsymbol{k}+\boldsymbol{P}/2}) f(\xi_{-\boldsymbol{k}+\boldsymbol{P}/2})}}{i\nu_\ell - \xi_{\boldsymbol{k}+\boldsymbol{P}/2} - \xi_{-\boldsymbol{k}+\boldsymbol{P}/2}}$$

$$\rho_{\mathrm{NSR}} = 2\sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) - \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)\frac{\partial \Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}{\partial \mu} e^{i\nu_\ell \delta} \equiv \rho_0 + \delta\rho$$

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In the BEC regime ($\mu < 0, |\mu| \gg T$) ➡ $f(\xi_{\boldsymbol{k}}) \simeq 0, \rho_0 \simeq 0$

$$\Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) \simeq -\sum_{\boldsymbol{k}} \frac{1}{i\nu_\ell - \frac{P^2}{4m} - \frac{k^2}{m} + 2\mu}$$

$$= \frac{m\Lambda}{2\pi^2} - \frac{m}{4\pi}\sqrt{m\left(\frac{P^2}{4m} - i\nu_\ell - 2\mu\right)}$$

$$\frac{\partial \Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}{\partial\mu} \simeq \frac{m^2}{4\pi}\frac{1}{\sqrt{m\left(\frac{P^2}{4m} - i\nu_\ell - 2\mu\right)}}$$

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$$\rho_{\mathrm{NSR}} \simeq -\frac{2}{\beta}\sum_{\boldsymbol{P},\ell} \frac{\mathcal{Z}_{\mathrm{b}}(\boldsymbol{P}, i\nu_\ell)}{i\nu_\ell - \frac{P^2}{4m} + 2\mu + E_{\mathrm{b}}} e^{i\nu_\ell \delta} \qquad \mathcal{Z}_{\mathrm{b}}(\boldsymbol{P}, i\nu_\ell) = \frac{1}{2}\left[1 + \frac{a_s^{-1}}{\sqrt{m\left(\frac{P^2}{4m} - i\nu_\ell - 2\mu\right)}}\right]$$

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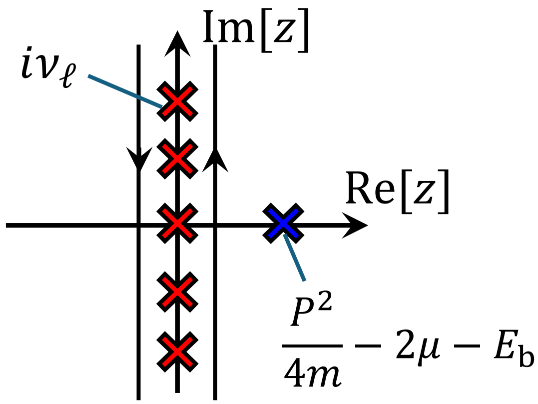
$$= -2\sum_{\boldsymbol{P}}\oint_C \frac{b(z)dz}{2\pi i}\frac{\mathcal{Z}_{\mathrm{b}}(\boldsymbol{P}, z)}{z - \frac{P^2}{4m} + 2\mu + E_{\mathrm{b}}} e^{z\delta}$$

$b(z) = \left(e^{\beta z} - 1\right)^{-1}$: Bose distribution function

$$= 2\sum_{\boldsymbol{P}} b\left(\frac{P^2}{4m} - 2\mu - E_{\mathrm{b}}\right)$$



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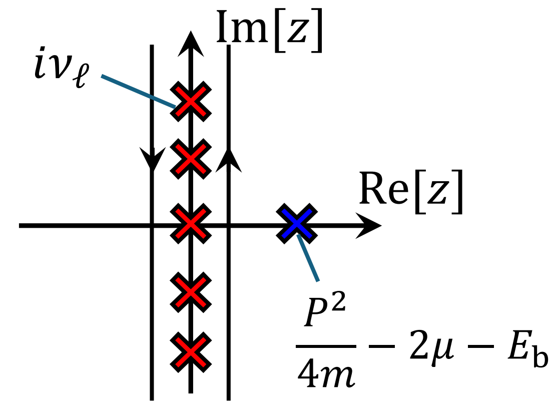

Renormalized gap equation: $\boxed{\frac{m}{4\pi a_s} = -\sum_{\boldsymbol{k}}\left[\frac{1 - 2f(E_{\boldsymbol{k}})}{2E_{\boldsymbol{k}}} - \frac{m}{k^2}\right]}$

In the BEC regime at $T_{\rm c}$ ($\mu < 0, |\mu| \gg T, \Delta \to 0$) ➡ $f(E_{\boldsymbol{k}}) \to f(\xi_{\boldsymbol{k}}) \simeq 0$

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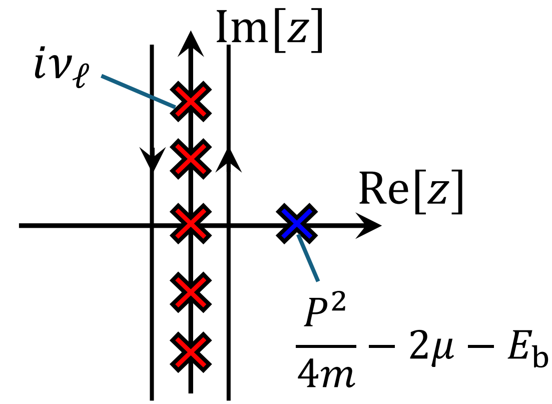

$$= 2\sum_{\boldsymbol{P}} b\left(\frac{P^2}{4m} - 2\mu - E_{\rm b}\right)$$

$$= 2\sum_{\boldsymbol{P}} \frac{1}{e^{\frac{P^2}{2(2m)T_{\rm c}}} - 1} = \underline{2\left(\frac{mT_{\rm c}}{\pi}\right)^{\frac{3}{2}} \zeta(3/2)}$$ ➡ **Molecular BEC temperature!**

Renormalized gap equation: $\boxed{\frac{m}{4\pi a_s} = -\sum_{\boldsymbol{k}}\left[\frac{1 - 2f(E_{\boldsymbol{k}})}{2E_{\boldsymbol{k}}} - \frac{m}{k^2}\right]}$

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Thouless criterion

At $T = T_{\mathrm{c}}$, the gap equation is equivalent to the gapless condition of T matrix pole

Renormalized gap equation at $\boldsymbol{T = T_{\mathrm{c}}}$

$$\boxed{\frac{m}{4\pi a_s} = -\sum_{\boldsymbol{k}} \left[\frac{1}{2\xi_{\boldsymbol{k}}} \tanh\left(\frac{\xi_{\boldsymbol{k}}}{2T_{\mathrm{c}}} \right) - \frac{m}{k^2} \right]} \quad \cdots (\bigstar)$$

Infrared divergence of in-medium T matrix at $\boldsymbol{T = T_{\mathrm{c}}}$

$$[T_{\mathrm{mb}}(\boldsymbol{P} = \boldsymbol{0}, i\nu_\ell = 0)]^{-1} = 0$$

$$= \frac{m}{4\pi a_s} + \Pi_{\mathrm{mb}}(\boldsymbol{P} = \boldsymbol{0}, i\nu_\ell = 0) - \sum_{\boldsymbol{k}} \frac{m}{k^2}$$

Thouless criterion

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Renormalized gap equation at $\boldsymbol{T = T_{\mathrm{c}}}$

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$$= \frac{m}{4\pi a_s} + \sum_{\boldsymbol{k}}\frac{1 - 2f(\xi_{\boldsymbol{k}})}{2\xi_{\boldsymbol{k}}} - \sum_{\boldsymbol{k}}\frac{m}{k^2} = (\bigstar)$$

Summary of NSR theory

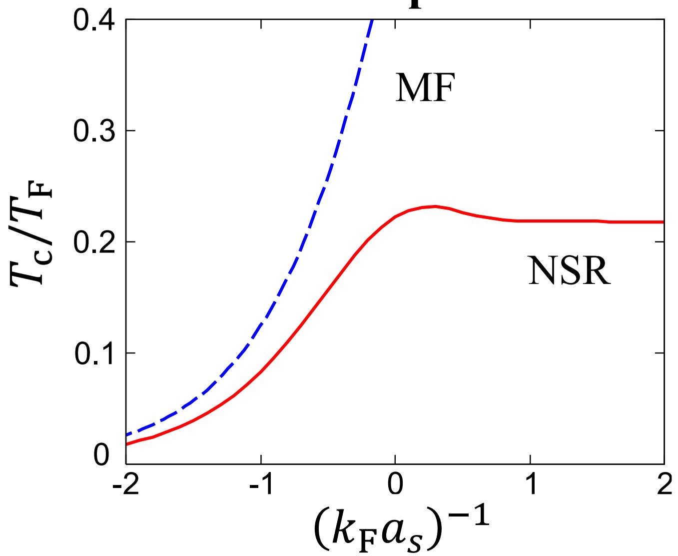
Particle number equation

$$\rho_{\rm NSR} = 2\sum_{\boldsymbol{k}} f(\xi_{\boldsymbol{k}}) - \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} T_{\rm mb}(\boldsymbol{P}, i\nu_\ell)\frac{\partial \Pi_{\rm mb}(\boldsymbol{P}, i\nu_\ell)}{\partial \mu} e^{i\nu_\ell \delta}$$

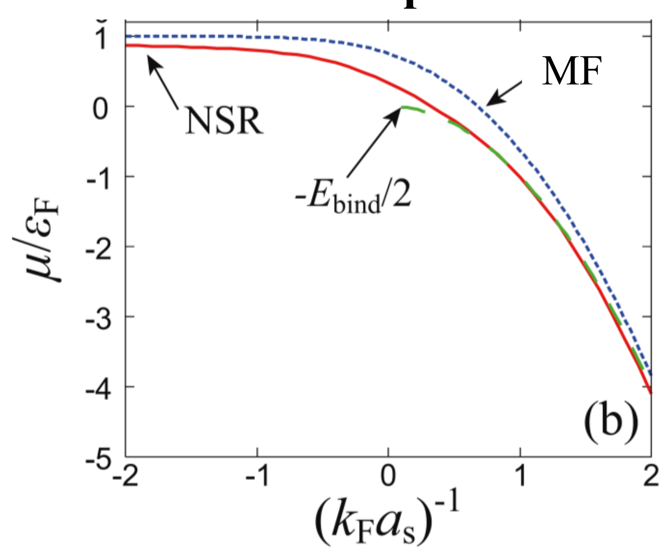
Thouless criterion

$$[T_{\rm mb}(\boldsymbol{P}=\boldsymbol{0}, i\nu_\ell = 0)]^{-1} = \frac{m}{4\pi a_s} + \sum_{\boldsymbol{k}}\left[\frac{1}{2\xi_{\boldsymbol{k}}}\tanh\left(\frac{\xi_{\boldsymbol{k}}}{2T_{\rm c}}\right) - \frac{m}{k^2}\right] = 0$$

Critical temperature



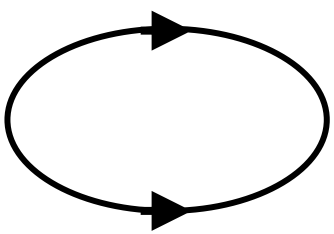
Chemical potential



Why does the ladder approximation work well?

Ladder diagram becomes dominant when
1. the system is dilute, 2. the interaction is short-range, or 3. bound state in free space is crucial

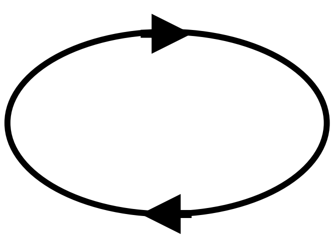
Particle-particle ladder



$$\sim \sum \frac{1 - f(\xi_p) - f(\xi_k)}{E - \xi_p - \xi_k} \quad \Rightarrow \quad \sim \Lambda_{\mathrm{cutoff}} \simeq \frac{1}{r_{\mathrm{eff}}}$$

$f(\xi_p) \to 0$

Particle-hole bubble



$$\sim \sum \frac{f(\xi_p) - f(\xi_k)}{E + \xi_p - \xi_k} \quad \Rightarrow \quad \sim 0$$

$f(\xi_p) \to 0$

ph bubble becomes important when the interaction is strong at low energies (e.g., Coulomb)

$f(\xi_p)$: Distribution function

Outline

- Why beyond mean-field approximation?
- Nozières-Schmitt-Rink theory
- Pseudogap and short-range correlations
- Short summary

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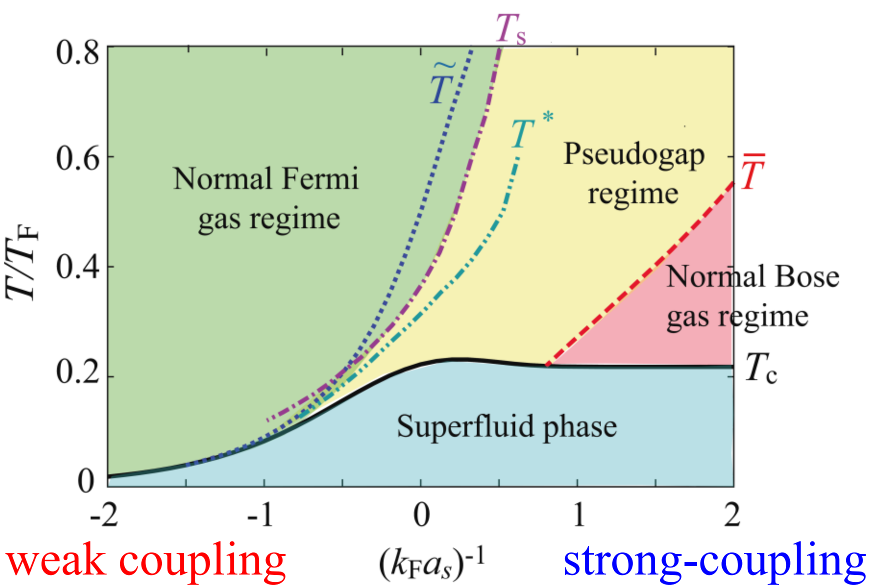
Strong pairing fluctuations and pseudogaps

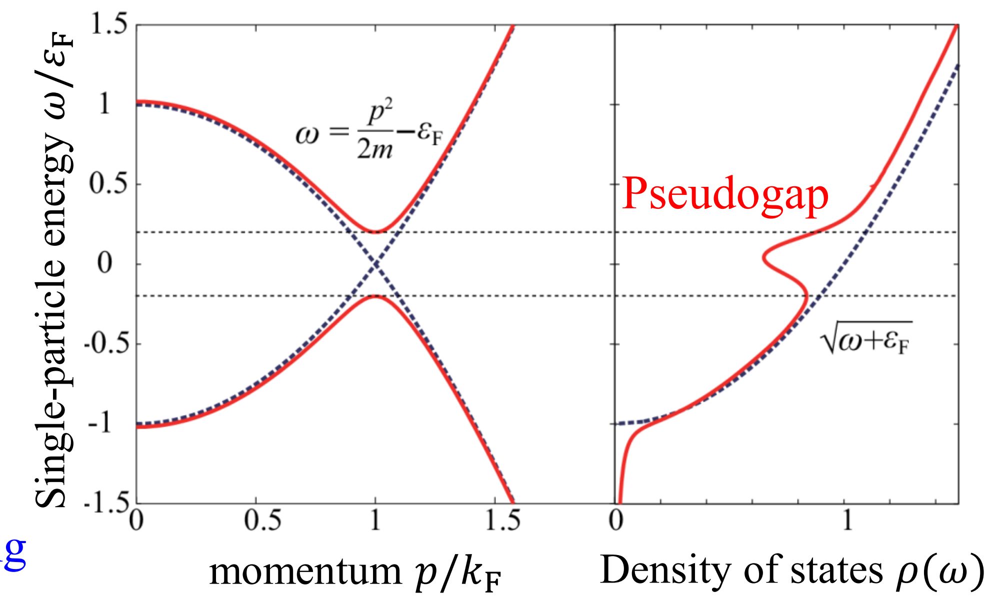
Review: Y. Ohashi, HT, and P. van Wyk, Prog. Part. Nucl. Phys. **111**, 103739 (2020).

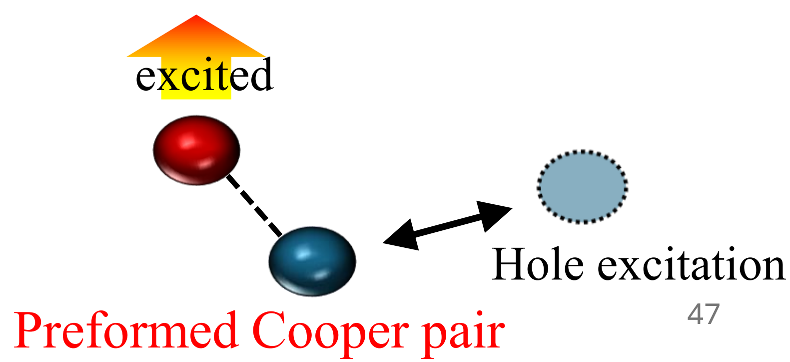
Phase diagram of from physical quantities



T_s: Spin susceptibility starts to drop
T^*: Density of states starts to show a dip
$\tilde{T}$: Specific heat becomes minimum

Single-particle dispersion and density of states

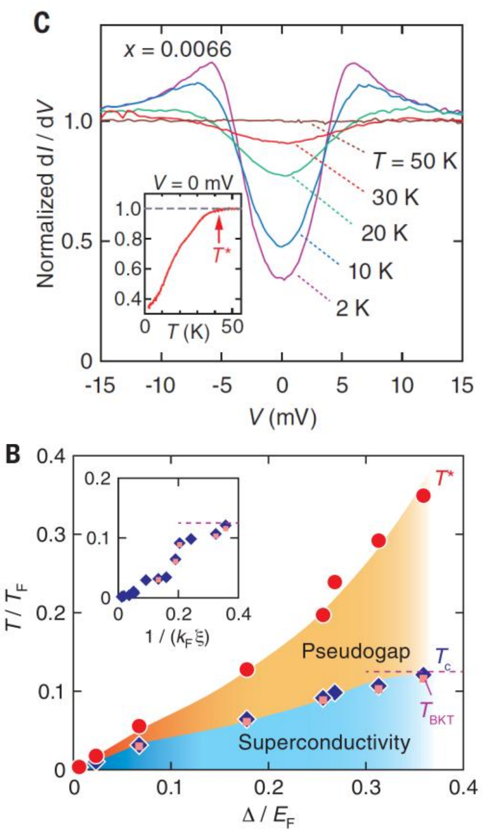




Pairing pseudogap effect across the different energy scales

Dip-like structure in density of states or level density even in the normal phase

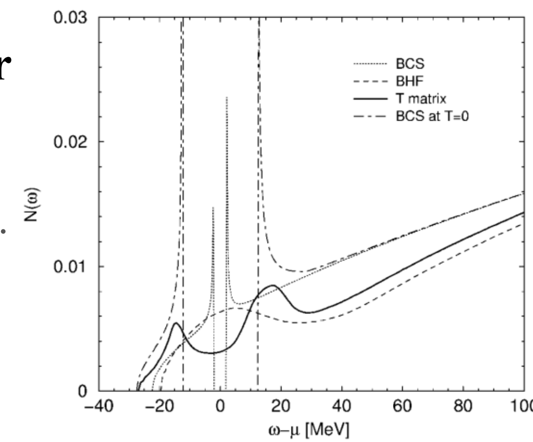
BCS-BEC crossover in Li_xZrNCl



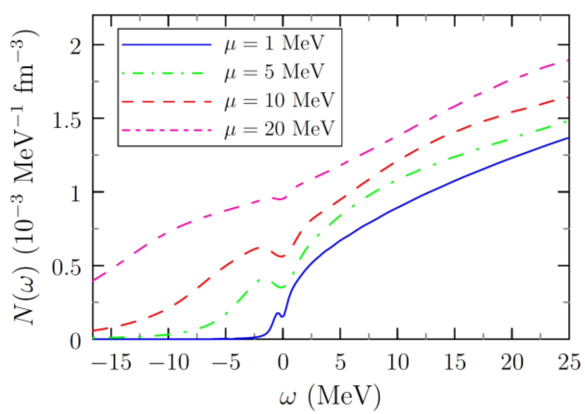
Y. Nakagawa, et al., Science **372**, 6538 (2021).

Nuclear matter

A. Schnell, et al., PRL **83**, 1926 (1999).

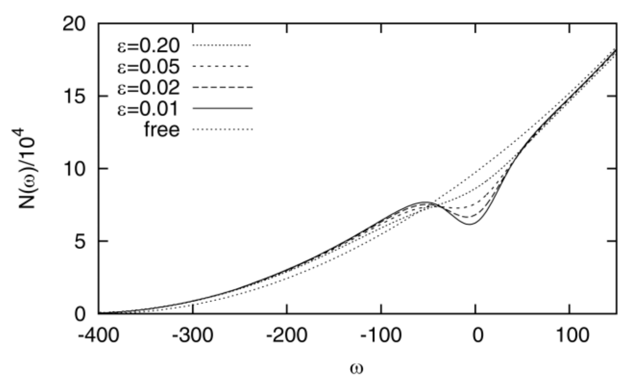


Neutron matter



D. Durel, et al., Universe **2020**, 6(11), 208

Color superconductivity



M. Kitazawa, et al., PRD **70**, 056003 (2004).

Extension of NSR: T-matrix approximation

$$\rho_{\mathrm{NSR}} = \frac{1}{\beta}\sum_{\boldsymbol{k},\sigma,\ell'} G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) - \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)\frac{\partial \Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}{\partial\mu} e^{i\nu_\ell\delta}$$

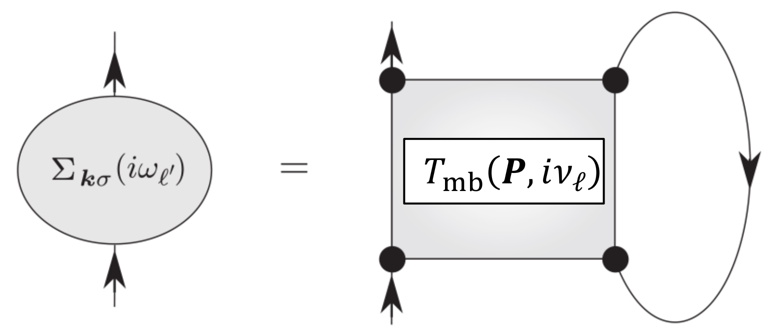
$$\boxed{\frac{\partial \Pi_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)}{\partial\mu} = -\frac{1}{\beta}\sum_{\boldsymbol{k},\sigma,\ell''}\left[G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell''})\right]^2 G^0_{\boldsymbol{P}-\boldsymbol{k}\bar{\sigma}}(i\nu_\ell - i\omega_{\ell''})}$$

$\bar{\sigma}$: opposite spin of σ

$$= \frac{1}{\beta}\sum_{\boldsymbol{k},\sigma,\ell'}\left[G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) + G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'})\Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'})G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'})\right]$$

Self-energy with *T*-matrix approximation

$$\Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) = \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)\, G^0_{\boldsymbol{P}-\boldsymbol{k}\bar{\sigma}}(i\nu_\ell - i\omega_{\ell'})$$



T-matrix approximation

NSR theory is obtained as truncation of **Dyson equation** with respect to Σ

➡ Include the self-energy such that the **Dyson equation** is satisfied

$$\rho_{\mathrm{NSR}} = \frac{1}{\beta} \sum_{\boldsymbol{k},\sigma,\ell'} \left[G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) + G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) \Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) \right]$$

➡
$$\rho_{\mathrm{TMA}} = \frac{1}{\beta} \sum_{\boldsymbol{k},\sigma,\ell'} \left[G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) + G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) \Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) G_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) \right]$$

$$\equiv \frac{1}{\beta} \sum_{\boldsymbol{k},\sigma,\ell'} G_{\boldsymbol{k}\sigma}(i\omega_{\ell'})$$

Dyson equation

$$G_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) = G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) + G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) \Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) G_{\boldsymbol{k}\sigma}(i\omega_{\ell'})$$

Single-particle spectral weight

Dyson equation

$$G_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) = G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) + G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'})\Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'})G_{\boldsymbol{k}\sigma}(i\omega_{\ell'})$$

$$\equiv \frac{1}{i\omega_{\ell'} - \xi_{\boldsymbol{k}} - \Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'})}$$

* This structure is absent in NSR due to the truncation of Σ

Spectral weight

$$A_{\boldsymbol{k}\sigma}(\omega) = \sum_{n,n'} \frac{e^{-\beta E_n} + e^{-\beta E_{n'}}}{Z} \langle \Psi_n | c_{\boldsymbol{k}\sigma} | \Psi_{n'} \rangle \langle \Psi_{n'} \left| c^\dagger_{\boldsymbol{k}\sigma} \right| \Psi_n \rangle \delta(\omega + E_n - E_{n'})$$

$* \; H|\Psi_n\rangle = E_n|\Psi_n\rangle$

$$= -\frac{1}{\pi}\mathrm{Im}\left[G_{\boldsymbol{k}\sigma}(i\omega_{\ell'} \to \omega + i\delta)\right] \quad \text{(analytic continuation)}$$

Single-particle spectral weight

Dyson equation

$$G_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) = G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) + G^0_{\boldsymbol{k}\sigma}(i\omega_{\ell'})\Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'})G_{\boldsymbol{k}\sigma}(i\omega_{\ell'})$$

$$\equiv \frac{1}{i\omega_{\ell'} - \xi_{\boldsymbol{k}} - \Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'})}$$

* This structure is absent in NSR due to the truncation of Σ

Spectral weight

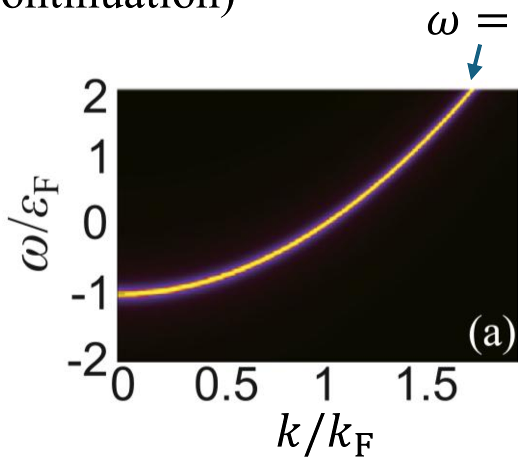
$$A_{\boldsymbol{k}\sigma}(\omega) = \sum_{n,n'} \frac{e^{-\beta E_n} + e^{-\beta E_{n'}}}{Z} \langle \Psi_n | c_{\boldsymbol{k}\sigma} | \Psi_{n'} \rangle \langle \Psi_{n'} \left| c^\dagger_{\boldsymbol{k}\sigma} \right| \Psi_n \rangle \delta(\omega + E_n - E_{n'})$$

$* \, H|\Psi_n\rangle = E_n|\Psi_n\rangle$

$$= -\frac{1}{\pi}\mathrm{Im}\left[G_{\boldsymbol{k}\sigma}(i\omega_{\ell'} \to \omega + i\delta)\right] \quad \text{(analytic continuation)}$$

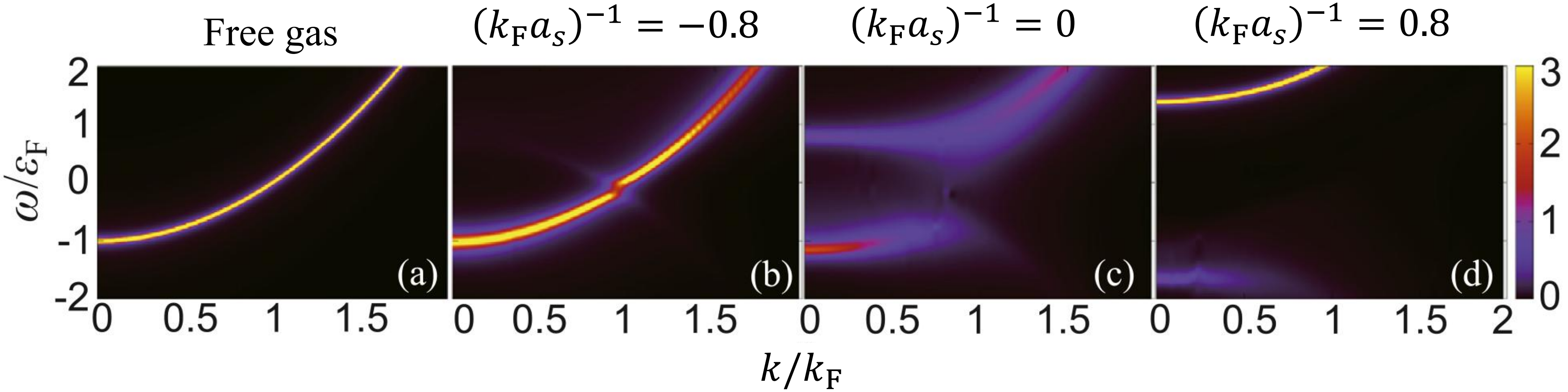
Non-interacting case (Σ = 0)

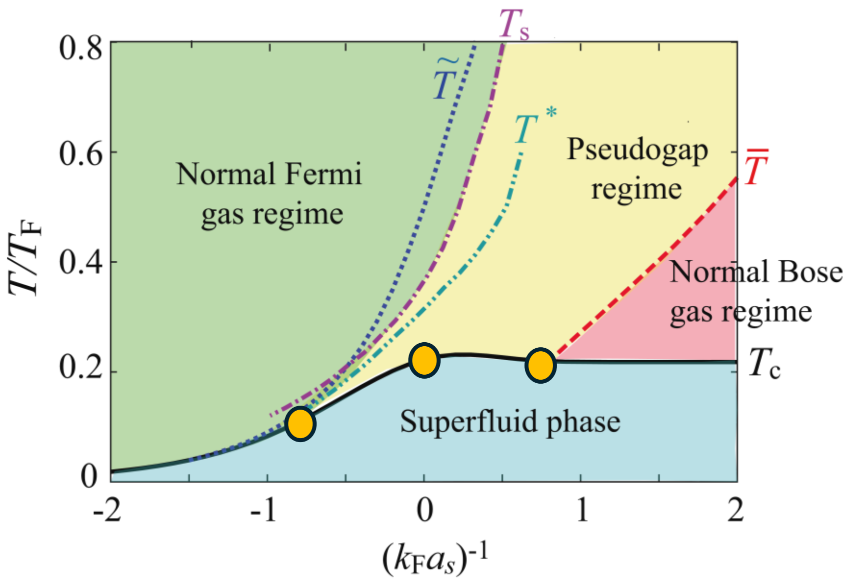
$$A_{\boldsymbol{k}\sigma}(\omega) = -\frac{1}{\pi}\mathrm{Im}\left[\frac{1}{\omega + i\delta - \xi_{\boldsymbol{k}}}\right] = \delta(\omega - \xi_{\boldsymbol{k}})$$



Single-particle excitation at T_c

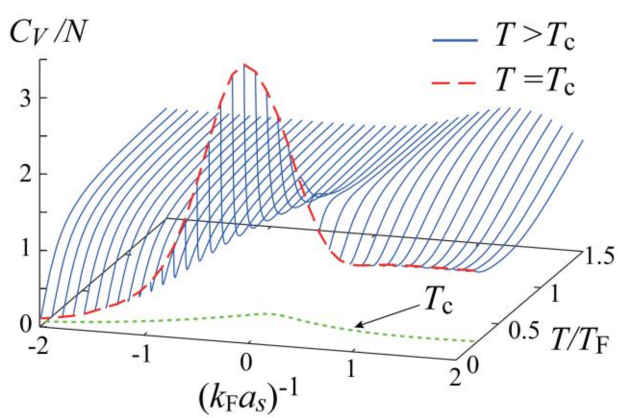
Even without pairing gap (i.e., $\Delta = 0$), single-particle excitation is gapped ➡**pseudogap**





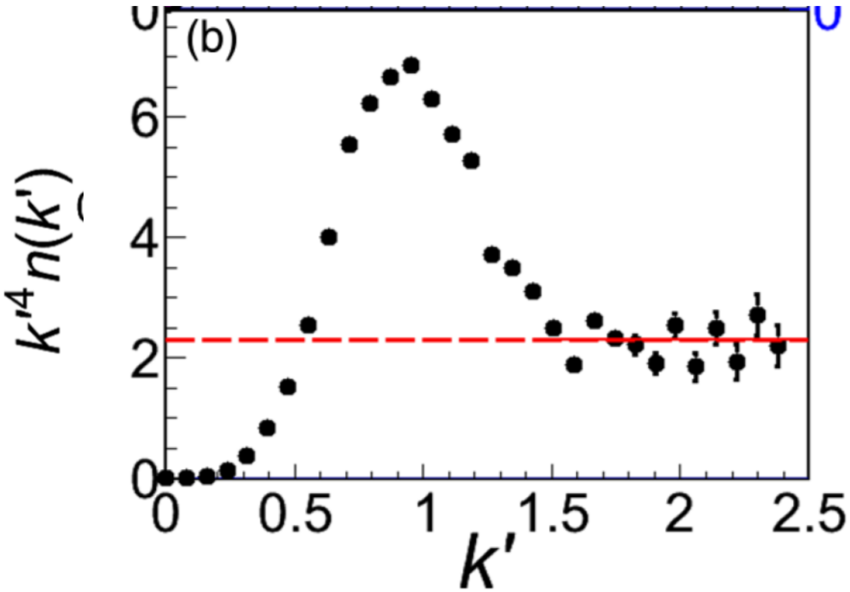
Pseudogap affects various observables

e.g., enhanced specific heat



P. van Wyk, et al., Phys. Rev. A **93**, 013621 (2016)

Relation to short-range correlations



High-momentum tail

O. Hen, et al., PRC **92**, 045205 (2015).

Relation to short-range correlations

$$\Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) = \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_{\ell})\, G^{0}_{\boldsymbol{P}-\boldsymbol{k}\bar{\sigma}}(i\nu_{\ell} - i\omega_{\ell'})$$

Relation to short-range correlations

$$\Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) = \frac{1}{\beta}\sum_{\boldsymbol{P},\ell} T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell)\, G^0_{\boldsymbol{P}-\boldsymbol{k}\bar{\sigma}}(i\nu_\ell - i\omega_{\ell'})$$

Spectral representation

$$T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell) = \int_{-\infty}^{\infty} d\Omega \frac{A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\nu_\ell - \Omega} \qquad A_{\mathrm{b}}(\boldsymbol{P},\Omega) = -\frac{1}{\pi}\mathrm{Im}[T_{\mathrm{mb}}(\boldsymbol{P}, i\nu_\ell \to \Omega + i\delta)]$$

$$= \frac{1}{\beta}\sum_{\boldsymbol{P},\ell}\int_{-\infty}^{\infty} d\Omega \frac{A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{(i\nu_\ell - \Omega)(i\nu_\ell - i\omega_{\ell'} - \xi_{\boldsymbol{P}-\boldsymbol{k}})}$$

$$= -\sum_{\boldsymbol{P}}\oint_C \frac{b(z)dz}{2\pi i}\int_{-\infty}^{\infty} d\Omega \frac{A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{(z - \Omega)(z - i\omega_{\ell'} - \xi_{\boldsymbol{P}-\boldsymbol{k}})}$$

$$= -\sum_{\boldsymbol{P}}\int_{-\infty}^{\infty} d\Omega \frac{b(\Omega)A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{\Omega - i\omega_{\ell'} - \xi_{\boldsymbol{P}-\boldsymbol{k}}} - \sum_{\boldsymbol{P}}\int_{-\infty}^{\infty} d\Omega \frac{b(\xi_{\boldsymbol{P}-\boldsymbol{k}} + i\omega_{\ell'})A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\omega_{\ell'} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Relation to short-range correlations

$$\Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\omega_{\ell'} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega} - \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{b(\xi_{\boldsymbol{P}-\boldsymbol{k}} + i\omega_{\ell'}) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\omega_{\ell'} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

$$b(\xi_{\boldsymbol{P}-\boldsymbol{k}} + i\omega_{\ell'}) = \frac{1}{e^{\beta\xi_{\boldsymbol{P}-\boldsymbol{k}} + i(2\ell'+1)} - 1} = -f(\xi_{\boldsymbol{P}-\boldsymbol{k}})$$

$$= \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\omega_{\ell'} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega} + \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{f(\xi_{\boldsymbol{P}-\boldsymbol{k}}) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\omega_{\ell'} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Relation to short-range correlations

$$\Sigma_{\boldsymbol{k}\sigma}(i\omega_{\ell'}) = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\omega_{\ell'} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega} - \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{b(\xi_{\boldsymbol{P}-\boldsymbol{k}} + i\omega_{\ell'}) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\omega_{\ell'} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

$$b(\xi_{\boldsymbol{P}-\boldsymbol{k}} + i\omega_{\ell'}) = \frac{1}{e^{\beta\xi_{\boldsymbol{P}-\boldsymbol{k}} + i(2\ell'+1)} - 1} = -f(\xi_{\boldsymbol{P}-\boldsymbol{k}})$$

$$= \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\omega_{\ell'} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega} + \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{f(\xi_{\boldsymbol{P}-\boldsymbol{k}}) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{i\omega_{\ell'} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Analytic continuation $(i\omega_{\ell'} \to \omega + i\delta)$

$$\Sigma_{\boldsymbol{k}\sigma}(\omega) = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{\omega + i\delta + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega} + \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{f(\xi_{\boldsymbol{P}-\boldsymbol{k}}) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{\omega + i\delta + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Short-range correlation **Brueckner Hartree-Fock term**

$$\xrightarrow{k\to\infty} \frac{\#}{k^2} \qquad\qquad \xrightarrow{k\to\infty} e^{-k^2/2mT}$$

Brueckner Hartree-Fock term

$$\Sigma_{\boldsymbol{k}\sigma}(\omega) = \Sigma_{\boldsymbol{k}\sigma}^{\mathrm{SRC}}(\omega) + \boxed{\Sigma_{\boldsymbol{k}\sigma}^{\mathrm{BHF}}(\omega)}$$

$$\Sigma_{\boldsymbol{k}\sigma}^{\mathrm{BHF}}(\omega) = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{f(\xi_{\boldsymbol{P}-\boldsymbol{k}}) A_{\mathrm{b}}(\boldsymbol{P}, \Omega)}{\omega + i\delta + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Brueckner Hartree-Fock term

$$\Sigma_{\boldsymbol{k}\sigma}(\omega) = \Sigma_{\boldsymbol{k}\sigma}^{\mathrm{SRC}}(\omega) + \boxed{\Sigma_{\boldsymbol{k}\sigma}^{\mathrm{BHF}}(\omega)}$$

$$\Sigma_{\boldsymbol{k}\sigma}^{\mathrm{BHF}}(\omega) = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{f(\xi_{\boldsymbol{P}-\boldsymbol{k}}) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{\omega + i\delta + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Single-particle potential (onshell) : Fermi-liquid correction

$$U_{\boldsymbol{k}\sigma} = \mathrm{Re}\left[\Sigma_{\boldsymbol{k}\sigma}^{\mathrm{BHF}}(\omega = \xi_{\boldsymbol{k}})\right] = \sum_{\boldsymbol{P}} P \int_{-\infty}^{\infty} d\Omega \frac{f(\xi_{\boldsymbol{P}-\boldsymbol{k}}) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{\xi_{\boldsymbol{k}} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Kramers-Kronig relation: $\mathrm{Re}[T_{\mathrm{mb}}(\boldsymbol{P}, \xi_{\boldsymbol{k}} + \xi_{\boldsymbol{P}-\boldsymbol{k}})] = \dfrac{1}{\pi} P \displaystyle\int_{-\infty}^{\infty} d\Omega \frac{\mathrm{Im}[T_{\mathrm{mb}}(\boldsymbol{P},\Omega)]}{\Omega - \xi_{\boldsymbol{k}} - \xi_{\boldsymbol{P}-\boldsymbol{k}}}$

Brueckner Hartree-Fock term

$$\Sigma_{\boldsymbol{k}\sigma}(\omega) = \Sigma^{\mathrm{SRC}}_{\boldsymbol{k}\sigma}(\omega) + \Sigma^{\mathrm{BHF}}_{\boldsymbol{k}\sigma}(\omega)$$

$$\Sigma^{\mathrm{BHF}}_{\boldsymbol{k}\sigma}(\omega) = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{f(\xi_{\boldsymbol{P}-\boldsymbol{k}}) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{\omega + i\delta + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Single-particle potential (onshell) : Fermi-liquid correction

$$U_{\boldsymbol{k}\sigma} = \mathrm{Re}\left[\Sigma^{\mathrm{BHF}}_{\boldsymbol{k}\sigma}(\omega = \xi_{\boldsymbol{k}})\right] = \sum_{\boldsymbol{P}} P \int_{-\infty}^{\infty} d\Omega \frac{f(\xi_{\boldsymbol{P}-\boldsymbol{k}}) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{\xi_{\boldsymbol{k}} + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Kramers-Kronig relation: $\mathrm{Re}[T_{\mathrm{mb}}(\boldsymbol{P}, \xi_{\boldsymbol{k}} + \xi_{\boldsymbol{P}-\boldsymbol{k}})] = \dfrac{1}{\pi} P \displaystyle\int_{-\infty}^{\infty} d\Omega \frac{\mathrm{Im}[T_{\mathrm{mb}}(\boldsymbol{P},\Omega)]}{\Omega - \xi_{\boldsymbol{k}} - \xi_{\boldsymbol{P}-\boldsymbol{k}}}$

Form of BHF single-particle potential often used in nuclear physics

$$U_{\boldsymbol{k}\sigma} = \sum_{\boldsymbol{P}} \mathrm{Re}[T_{\mathrm{mb}}(\boldsymbol{P}, \xi_{\boldsymbol{k}} + \xi_{\boldsymbol{P}-\boldsymbol{k}})] f(\xi_{\boldsymbol{P}-\boldsymbol{k}}) \xrightarrow{T\to 0} \sum_{|\boldsymbol{P}|\le k_{\mathrm{F}}} \mathrm{Re}[T_{\mathrm{mb}}(\boldsymbol{P}+\boldsymbol{k}, \xi_{\boldsymbol{k}+\boldsymbol{P}} + \xi_{\boldsymbol{P}})]$$

Short-range correlation term

$$\Sigma_{\boldsymbol{k}\sigma}(\omega) = \boxed{\Sigma^{\mathrm{SRC}}_{\boldsymbol{k}\sigma}(\omega)} + \Sigma^{\mathrm{BHF}}_{\boldsymbol{k}\sigma}(\omega)$$

$$\Sigma^{\mathrm{SRC}}_{\boldsymbol{k}\sigma}(\omega) = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{\omega + i\delta + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Short-range correlation term

$$\Sigma_{\boldsymbol{k}\sigma}(\omega) = \boxed{\Sigma^{\mathrm{SRC}}_{\boldsymbol{k}\sigma}(\omega)} + \Sigma^{\mathrm{BHF}}_{\boldsymbol{k}\sigma}(\omega)$$

$$\Sigma^{\mathrm{SRC}}_{\boldsymbol{k}\sigma}(\omega) = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega \frac{b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)}{\omega + i\delta + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega}$$

Assumption:

The $(\boldsymbol{P},\Omega)$ dependence of $(\omega + i\delta + \xi_{\boldsymbol{P}-\boldsymbol{k}} - \Omega)^{-1}$ is much weaker than $b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)$

$$\Sigma^{\mathrm{SRC}}_{\boldsymbol{k}\sigma}(\omega) \simeq \frac{1}{\omega + i\delta + \xi_{-\boldsymbol{k}}} \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega\, b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)$$

$$\equiv \frac{\Delta^2_{\mathrm{pg}}}{\omega + i\delta + \xi_{-\boldsymbol{k}}}$$

Pseudogap parameter: $\Delta^2_{\mathrm{pg}} = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega\, b(\Omega) A_{\mathrm{b}}(\boldsymbol{P},\Omega)$

Short-range correlation term

Approximate SRC term: $$\Sigma^{\mathrm{SRC}}_{\boldsymbol{k}\sigma}(\omega) \simeq \frac{\Delta^2_{\mathrm{pg}}}{\omega + i\delta + \xi_{-\boldsymbol{k}}}$$

Pseudogap parameter: $$\Delta^2_{\mathrm{pg}} = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega\, b(\Omega) A_{\mathrm{b}}(\boldsymbol{P}, \Omega)$$

Short-range correlation term

Approximate SRC term: $\Sigma^{\mathrm{SRC}}_{\boldsymbol{k}\sigma}(\omega) \simeq \dfrac{\Delta_{\mathrm{pg}}^2}{\omega + i\delta + \xi_{-\boldsymbol{k}}}$

Pseudogap parameter: $\Delta_{\mathrm{pg}}^2 = \sum_{\boldsymbol{P}} \int_{-\infty}^{\infty} d\Omega\, b(\Omega) A_{\mathrm{b}}(\boldsymbol{P}, \Omega)$

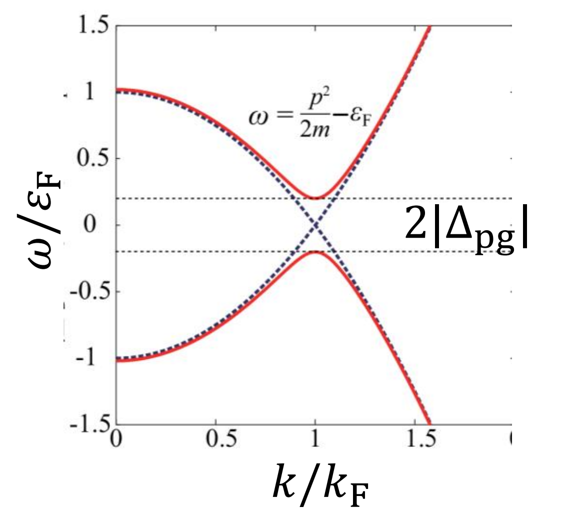
$$G_{\boldsymbol{k}\sigma}(\omega) \simeq \frac{1}{\omega + i\delta - \xi_{\boldsymbol{k}} - \dfrac{\Delta_{\mathrm{pg}}^2}{\omega + i\delta + \xi_{\boldsymbol{k}}}} = \frac{\omega + i\delta + \xi_{\boldsymbol{k}}}{(\omega + i\delta - \xi_{\boldsymbol{k}})^2 - \Delta_{\mathrm{pg}}^2}$$

$$= \frac{u_{\boldsymbol{k},\mathrm{pg}}^2}{\omega + i\delta - \sqrt{\xi_{\boldsymbol{k}}^2 + \Delta_{\mathrm{pg}}^2}} + \frac{v_{\boldsymbol{k},\mathrm{pg}}^2}{\omega + i\delta + \sqrt{\xi_{\boldsymbol{k}}^2 + \Delta_{\mathrm{pg}}^2}}$$

$$u_{\boldsymbol{k},\mathrm{pg}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\boldsymbol{k}}}{\sqrt{\xi_{\boldsymbol{k}}^2 + \Delta_{\mathrm{pg}}^2}}\right)$$

$$v_{\boldsymbol{k},\mathrm{pg}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\boldsymbol{k}}}{\sqrt{\xi_{\boldsymbol{k}}^2 + \Delta_{\mathrm{pg}}^2}}\right)$$

$A_{\boldsymbol{k}\sigma}(\omega) = -\dfrac{1}{\pi}\mathrm{Im}[G_{\boldsymbol{k}\sigma}(\omega)]$
exhibits gapped structure!



High-momentum tail

$$n_{\boldsymbol{k}\sigma} = \left\langle c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} \right\rangle \equiv G_{\boldsymbol{k}\sigma}(\tau = -\delta)$$

High-momentum tail

$$
\begin{aligned}
n_{\boldsymbol{k}\sigma} = \left\langle c_{\boldsymbol{k}\sigma}^{\dagger} c_{\boldsymbol{k}\sigma} \right\rangle &\equiv G_{\boldsymbol{k}\sigma}(\tau = -\delta) \\
&= \frac{1}{\beta} \sum_{\ell} G_{\boldsymbol{k}\sigma}(i\omega_\ell) e^{i\omega_\ell \delta} \\
&= -\oint_C \frac{f(z)dz}{2\pi i} f(z) e^{z\delta} \int_{-\infty}^{\infty} d\omega \frac{A_{\boldsymbol{k}\sigma}(\omega)}{z-\omega} \\
&= \int_{-\infty}^{\infty} d\omega\, f(\omega) A_{\boldsymbol{k}\sigma}(\omega)
\end{aligned}
$$

High-momentum tail

$$
\begin{aligned}
n_{\boldsymbol{k}\sigma} = \left\langle c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} \right\rangle &\equiv G_{\boldsymbol{k}\sigma}(\tau = -\delta) \\
&= \frac{1}{\beta} \sum_{\ell} G_{\boldsymbol{k}\sigma}(i\omega_\ell) e^{i\omega_\ell \delta} \\
&= -\oint_C \frac{f(z)dz}{2\pi i} f(z) e^{z\delta} \int_{-\infty}^{\infty} d\omega \frac{A_{\boldsymbol{k}\sigma}(\omega)}{z - \omega} \\
&= \int_{-\infty}^{\infty} d\omega\, f(\omega) A_{\boldsymbol{k}\sigma}(\omega)
\end{aligned}
$$

Pseudogapped spectrum due to SRC term

$$
A_{\boldsymbol{k}\sigma}(\omega) \simeq u^2_{\boldsymbol{k},\mathrm{pg}} \delta\left(\omega - \sqrt{\xi^2_{\boldsymbol{k}} + \Delta^2_{\mathrm{pg}}}\right) + v^2_{\boldsymbol{k},\mathrm{pg}} \delta\left(\omega + \sqrt{\xi^2_{\boldsymbol{k}} + \Delta^2_{\mathrm{pg}}}\right)
$$

High-momentum tail above T_{c}

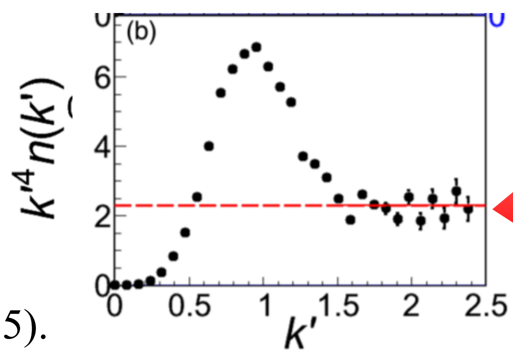
$$n_{\boldsymbol{k}\sigma} = \int_{-\infty}^{\infty} d\omega\, f(\omega) A_{\boldsymbol{k}\sigma}(\omega)$$

$$\simeq u_{\boldsymbol{k},\mathrm{pg}}^2 f\left(\sqrt{\xi_{\boldsymbol{k}}^2 + \Delta_{\mathrm{pg}}^2}\right) + v_{\boldsymbol{k},\mathrm{pg}}^2 f\left(-\sqrt{\xi_{\boldsymbol{k}}^2 + \Delta_{\mathrm{pg}}^2}\right)$$

Taking $|\boldsymbol{k}| \to \infty$ $\quad f\left(\sqrt{\xi_{\boldsymbol{k}}^2 + \Delta_{\mathrm{pg}}^2}\right) \to 0$

$$\simeq v_{\boldsymbol{k},\mathrm{pg}}^2 \equiv \frac{1}{2}\left(1 - \frac{\xi_{\boldsymbol{k}}}{\sqrt{\xi_{\boldsymbol{k}}^2 + \Delta_{\mathrm{pg}}^2}}\right) \xrightarrow{k\to\infty} \frac{m^2 |\Delta_{\mathrm{pg}}|^2}{k^4}$$

High-momentum tail $\simeq$ pseudogap size



O. Hen, et al., PRC **92**, 045205 (2015).

Outline

- Why beyond mean-field approximation?
- Nozières-Schmitt-Rink theory
- Pseudogap and short-range correlations
- Short summary

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- Why beyond mean-field approximation?
- Nozières-Schmitt-Rink theory
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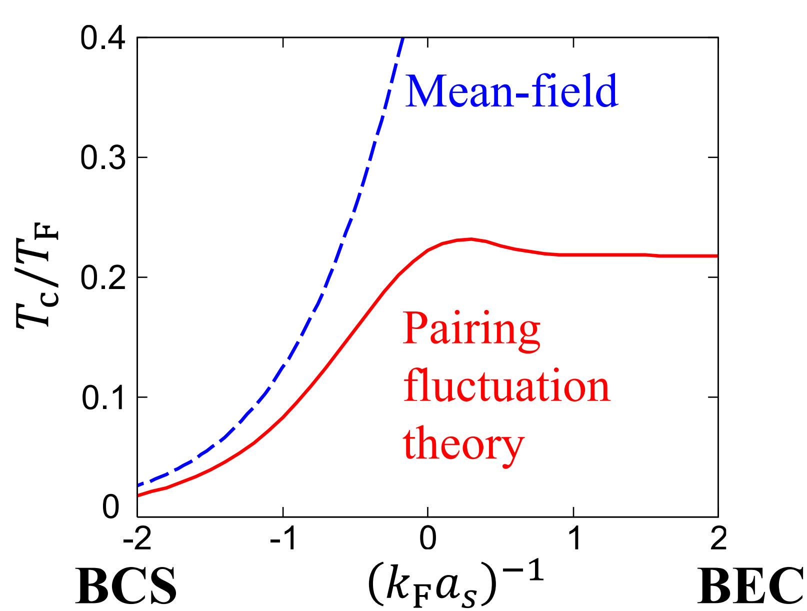
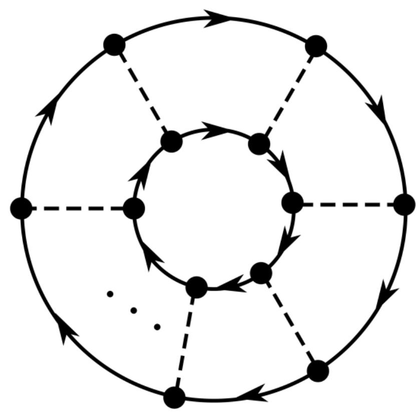
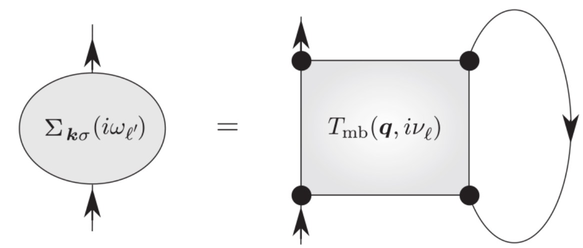
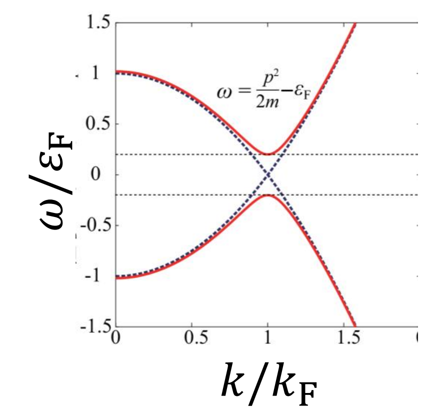
Short summary of Part 5

- Pairing fluctuation effect is important to describe the critical temperature curve throughout the BCS-BEC crossover.
- The Nozières-Schmitt-Rink theory and its extension have been presented.
- As an example of pairing fluctuation effect, the relationship between the pseudogap and the short-range correlations has been discussed.







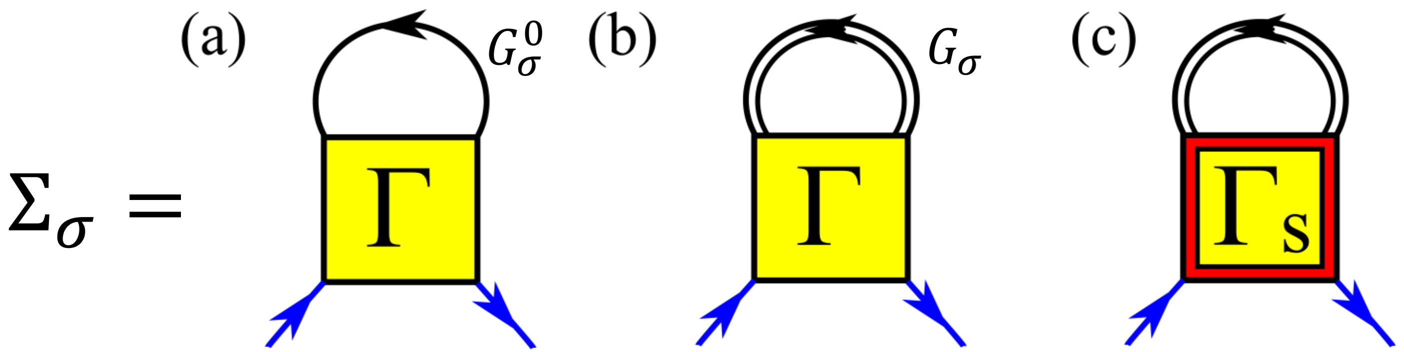


Appendix

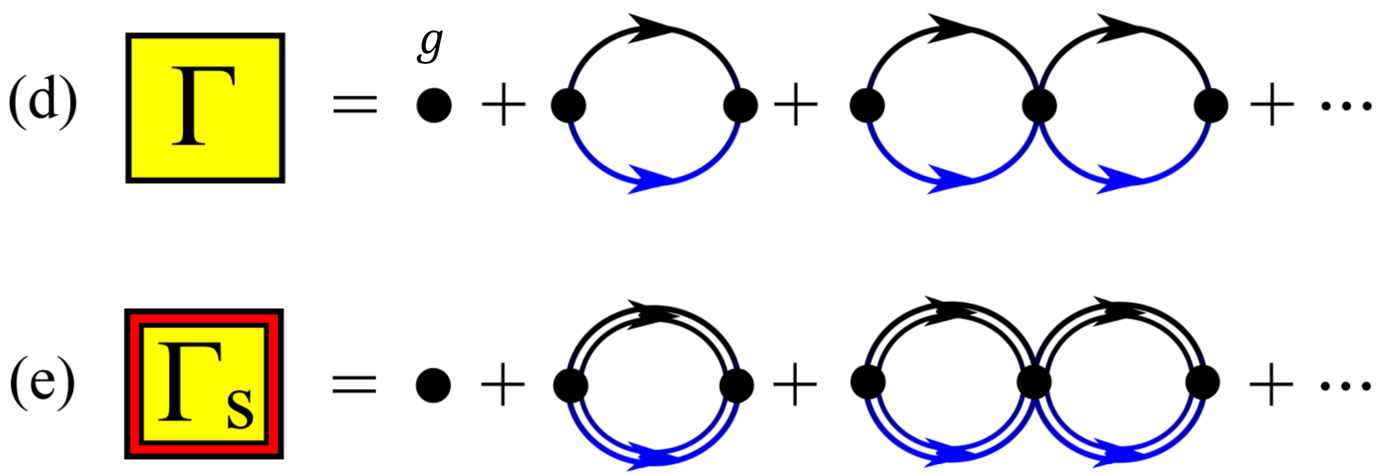
Variety of diagrammatic approaches

Self-energy diagrams

T-matrix approx. (TMA) | Extended TMA (ETMA) | Self-consistent TMA (SCTMA, LW)



Many-body *T*-matrix



Variety of diagrammatic approaches